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LAMINAR BOUNDARY-LAYER FLOW NEAR SEPARATION WITH AND WITHOUT SUCTION

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CONTENTS

Numerical solutions of the laminar boundary-layer equation for the mainstream velocity $U = U_0(1-\frac{1}{8}x)$ without suction have been obtained by Hartree and Leigh, and the solutions have suggested that a singularity is present at the separation point. Assuming the existence of

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this singularity, Goldstein developed an asymptotic solution in the upstream neighbourhood of separation, but his solution required that a certain integral condition must be satisfied. Stewartson extended this asymptotic solution so as to be independent of any integral condition. Jones and Leigh have compared the numerical and asymptotic solutions and have found satisfactory agreement between them.

In part I of this paper the work of Goldstein and Stewartson has been extended to include suction through a porous surface. The stream function ψ_1 is expanded in a series of the type

$$\psi_1 = 2^{\frac{3}{4}}\xi^3 \sum_{r=0}^{6} \xi^r f_r(\eta) + 2^{\frac{3}{4}}\xi^8 \ln \xi [F_5(\eta) + \xi F_6(\eta)] + O(\xi^{10} \ln \xi),$$

where $\xi = x_1^{\frac{1}{4}}$, $\eta = y_1/2^{\frac{1}{4}}x_1^{\frac{1}{4}}$ and (x_1, y_1) are non-dimensional distances measured from the separation point. Analytical solutions for the functions $f_r(\eta)$ $(r = 0, 1, \ldots, 5)$ have been obtained and the solutions for $r = 0, 1, \ldots, 4$ reduce to those given by Goldstein in the case of zero suction. The solution for $f_5(\eta)$ without suction was confirmed by comparison with the numerical work of Jones, and corrections were made to his values for two constants. The solution for $f_6(\eta)$ without suction was next considered so as to show that Goldstein's condition is not satisfied. This condition required the vanishing of a certain integral estimated by Jones at $(-4 \pm 4)\,\alpha_1^6$; its value is now found to be $(-8{\cdot}62 \pm 0{\cdot}01)\,\alpha_1^6$. Following Stewartson, solutions for the functions $F_5(\eta)$ and $F_6(\eta)$ are given. Numerical expansions for the skin friction and the velocity distribution near to separation have been obtained. Numerical tables are given for the functions $f_3(\eta)$ and $f_4(\eta)$ and their derivatives which are required for the computation of the velocity distribution.

In part II there is developed a numerical method, suitable for an automatic computer, by which the velocity distributions at all cross-sections to separation can be obtained from that at the leading edge. In this method Görtler's transformation is applied to the boundary-layer equations and then, by means of the Hartree–Womersley approximation, derivatives are replaced by differences. The resulting simultaneous equations are solved by an iterative procedure which involves the inversion of matrices. The program has been written so that given a general external velocity distribution and velocity of suction only a few specified subroutines are required.

By this method, the boundary-layer flow was computed for the mainstream velocity $U = U_0 \sin x$ (corresponding to potential flow past a circular cylinder) and a certain constant velocity of suction. Tables have been included showing the velocity distributions at selected cross-sections and giving the skin friction, displacement thickness and momentum thickness. The position of separation obtained was 114·7° from the forward stagnation point, whereas for the same suction velocity, Bussman & Ulrich gave 120·9° using a series expansion. The difference between these values was discussed and the former shown to be accurate. Near separation similar behaviour to that found by Hartree and Leigh was experienced, thus confirming the existence of a singularity at separation. The numerical results were compared with the solution given in part I and excellent agreement was obtained. The functions $f_1(\eta) \ldots f_4(\eta)$ depend on a parameter α_1, which was determined by comparing the numerical results with the asymptotic expressions for the skin friction and the velocity distribution near to separation. Both methods gave $\alpha_1 \simeq 0{\cdot}555$.

The work was repeated for the same mainstream flow $U = U_0 \sin x$ without suction. The position of separation in this case was 104·45° and $\alpha_1 \simeq 0{\cdot}676$. (Leigh obtained $\alpha_1 \simeq 0{\cdot}492$ for the mainstream flow $U = U_0(1 - \frac{1}{8}x)$ without suction.)

A range of solutions of the equation of similar profiles has also been obtained. In particular, the curve which divides the wholly forward flows from those with backflow is shown. The separation profiles for the two cases of potential flow past a circular cylinder have been compared with corresponding solutions of the equation of similar profiles.

Fuller details of the numerical results, giving the velocity profiles at different cross-sections for both flows past a circular cylinder and the solutions of the equation of similar profiles, are contained in the author's Ph.D. thesis at Manchester University.

I. A SOLUTION OF THE LAMINAR BOUNDARY-LAYER EQUATIONS NEAR THE SEPARATION POINT

1. Introduction

The equation of momentum for steady, two-dimensional, incompressible laminar boundary-layer flow is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$=U\frac{\mathrm{d}U}{\mathrm{d}x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3}$$

where x is the distance measured along the surface, y is the distance normal to the surface, u and v are the velocity components in the directions of x increasing and y increasing respectively, ρ the density, ν the viscosity, $\mathrm{d}p/\mathrm{d}x$ is the pressure gradient and $U(x)$ the mainstream velocity.

The equation of continuity can be replaced by the introduction of the stream function ψ such that

$$u=\frac{\partial \psi}{\partial y}, \quad v=-\frac{\partial \psi}{\partial x}. \tag{4}$$

The boundary conditions are:

(i) at the surface $y=0$

$$u=0, \quad v=-(U_0\nu/l)^{\frac{1}{2}}v_s(x), \tag{5}$$

where l is a typical length and U_0 a typical velocity of the system and $v_s(x)$ the given non-dimensional velocity of suction;

(ii) as $y\to\infty$, u tends continuously to the mainstream velocity so that

$$u\to U(x); \qquad \partial^r u/\partial y^r\to 0 \quad (r\geqslant 1). \tag{6}$$

If at the surface $y=0$

$$\partial u/\partial y=0 \tag{7}$$

separation occurs. The equations may be made non-dimensional by writing

$$x'=\frac{x}{l}, \quad y'=\frac{R^{\frac{1}{2}}y}{l}, \quad u'=\frac{u}{U_0}, \quad v'=\frac{R^{\frac{1}{2}}v}{U_0}, \quad p'=\frac{p}{\rho U_0^2}, \tag{8}$$

where R is the Reynolds number U_0l/ν. Assume that the velocity profile at $x'=0$ is given by

$$u'(0,y')=a_1y'+a_2y'^2+a_3y'^3+\dots \tag{9}$$

and the pressure gradient by

$$-\mathrm{d}p'/\mathrm{d}x'=p_0+p_1x'+p_2x'^2+\dots \tag{10}$$

and the velocity of suction by

$$v'(x',0)=-v_s(x)=-(v_0+v_1x'+v_2x'^2+\dots). \tag{11}$$

The non-dimensional stream-function ψ' must satisfy the equation of motion

$$\frac{\partial^3\psi'}{\partial y'^3}=\frac{\partial\psi'}{\partial y'}\frac{\partial^2\psi'}{\partial x'\partial y'}-\frac{\partial\psi'}{\partial x'}\frac{\partial^2\psi'}{\partial y'^2}+\frac{\mathrm{d}p'}{\mathrm{d}x'} \tag{12}$$

and the boundary conditions

$$\left.\begin{aligned} \frac{\partial\psi'}{\partial x'} = v_s(x), \quad \frac{\partial\psi'}{\partial y'} = 0 \quad \text{at} \quad y' = 0, \\ \frac{\partial\psi'}{\partial y'} = u'(0, y') \quad \text{at} \quad x' = 0. \end{aligned}\right\} \tag{13}$$

If there is no singularity at the initial section ψ' can be expanded as a double-power series in x' and y' and certain coefficients of this series will be given by the conditions (13). On equating coefficients in equation (12), relations are found between the coefficients a, p, and v of equations (9), (10) and (11).

If $a_1 \neq 0$, the coefficients a_1, a_4, a_7 may be taken to be independent and the other coefficients a_2, a_3, a_5, ... are determined in terms of them by

$$\left.\begin{aligned} -2!\,a_2 &= p_0 + a_1 v_0, \\ 3!\,a_3 &= p_0 v_0 + a_1 v_0^2, \\ -5!\,a_5 &= 2a_1 p_1 + 3\,.\,4!\,a_4 v_0 + 2p_0 v_0^3 + 2a_1 v_0^4 + 2a_1^2 v_1. \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{14}$$

When $a_1 = 0$, only the coefficients $a_8, a_{12}, \ldots$ are independent and the remaining a's are given by

$$\left.\begin{aligned} -2!\,a_2 &= p_0, \\ 3!\,a_3 &= p_0 v_0, \\ -4!\,a_4 &= p_0 v_0^2, \\ 5!\,a_5 &= p_0 v_0^3, \\ -6!\,a_6 &= -2p_0 p_1 + p_0 v_0^4, \\ 7!\,a_7 &= -7p_0 p_1 v_0 + p_0 v_0^5 - 5p_0^2 v_1. \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{15}$$

The conditions (14) and (15) have been given by Rheinboldt (1956). If these conditions are not satisfied, there is an algebraic singularity at $x = 0$. The problem of finding the behaviour of the solution at points close to the initial section when $a_1 \neq 0$ and not all the conditions (14) are satisfied has been studied by Goldstein (1930) for flow over an impermeable surface, and this has been extended by Rheinboldt (1956) to include suction or blowing.

The solution for flow over an impermeable surface near a position of separation (i.e. when $a_1 = 0$) where not all the conditions (15) are satisfied has been considered by Goldstein (1948) and Stewartson (1958). This work will be extended to include suction or blowing.

Using the same method as Goldstein (1948), we assume that there exists a singularity at the separation point and put the variables into non-dimensional form in the following way. Let x_S, U_S and $\mathrm{d}U_S/\mathrm{d}x$ be the values of x, U and $\mathrm{d}U/\mathrm{d}x$ at separation. Then for the representative length and the Reynolds number take

$$l = -\frac{U_S}{\mathrm{d}U_S/\mathrm{d}x}, \quad R = \frac{U_S l}{\nu}, \tag{16}$$

and for non-dimensional distances take

$$x_1 = \frac{x_S - x}{l}, \quad y_1 = \frac{R^{\frac{1}{2}} y}{l}, \tag{17}$$

and also take $$u_1 = \frac{u}{U_S}, \quad v_1 = \frac{R^{\frac{1}{2}} v}{U_S}, \quad U_1 = \frac{U}{U_S}, \quad p_1 = \frac{p}{\rho U_S^2}, \quad \psi_1 = \frac{R^{\frac{1}{2}} \psi}{l U_S}. \tag{18}$$

Equations (1), (2) and (4) become

$$\left.\begin{aligned} -u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} &= \frac{\mathrm{d}p_1}{\mathrm{d}x_1} + \frac{\partial^2 u_1}{\partial y_1^2}, \\ u_1 = \frac{\partial \psi_1}{\partial y_1}, \quad v_1 &= \frac{\partial \psi_1}{\partial x_1}, \\ \frac{\mathrm{d}p_1}{\mathrm{d}x_1} &= -U_1 \frac{\mathrm{d}U_1}{\mathrm{d}x_1}. \end{aligned}\right\} \tag{19}$$

At separation, $$\left(\frac{\mathrm{d}p_1}{\mathrm{d}x_1}\right)_{x_1=0} = \left(-U_1 \frac{\mathrm{d}U_1}{\mathrm{d}x_1}\right)_{x_1=0} = -1,$$

so that the pressure gradient can be written

$$\frac{\mathrm{d}p_1}{\mathrm{d}x_1} = -(1 + P_1 x_1 + P_2 x_1^2 + \ldots). \tag{20}$$

The solution is found by taking

$$\xi = x_1^{\frac{1}{4}}, \quad \eta = \frac{y_1}{2^{\frac{1}{2}} x_1^{\frac{1}{4}}} \tag{21}$$

and $$\psi_1 = 2^{\frac{3}{2}} \xi^3 f(\xi, \eta). \tag{22}$$

Goldstein assumed that the first condition in (15), namely $2a_2 = -p_0$, is satisfied, and from equation (20) $p_0 = -1$ so that $a_2 = \frac{1}{2}$. As $\partial u_1/\partial y_1 = 0$, the boundary condition

$$u_1 = \tfrac{1}{2} y_1^2 + \sum_{r=3}^{\infty} a_r y_1^r \quad \text{at} \quad x_1 = 0 \tag{23}$$

was taken. In equation (22) he took

$$f(\xi, \eta) = \sum_{r=0}^{\infty} f_r(\eta)\, \xi^r \tag{24}$$

and as a result found that $$\left(\frac{\partial u_1}{\partial y_1}\right)_{y_1=0} = 2^{\frac{3}{2}} \sum_{r=1}^{\infty} \alpha_r x_1^{\frac{1}{4}(r+1)}, \tag{25}$$

where $$\alpha_r = f_r''(0).$$

By substituting (24) in the equation of motion, $f_r(\eta)$ can be determined. Goldstein reasoned that all the α_r in (25) can be found in terms of α_1 and P_r and that α_1 is probably determined by the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$. However, it is necessary that a certain integral (Goldstein 1948, equation (108)), arising from ensuring $f_6(\eta)$ is not exponentially large, be zero. This integral has now been shown to be non-zero (see §4 below).

Stewartson overcame this difficulty by introducing logarithmic terms into the expansion for ψ_1 in equation (22). By trial, it is found that the most general form for ψ_1 is

$$\psi_1 = 2^{\frac{3}{2}}\xi^3 \sum_{r=0}^{6} f_r(\eta)\,\xi^r + 2^{\frac{3}{2}}\xi^8 \ln\xi[F_5(\eta)+\xi F_6(\eta)] + O(\xi^{10}\ln\xi), \tag{26}$$

which gives
$$u_1 = 2\xi^2 \sum_{r=0}^{6} f'_r(\eta)\xi^r + 2\xi^7 \ln\xi[F'_5(\eta)+\xi F'_6(\eta)] + O(\xi^9\ln\xi). \tag{27}$$

It follows that the assumption (23) is invalid since u_1 contains terms that tend to infinity as $x_1 \to 0$. However, this form for ψ_1 resolves the difficulty over the integral because $F_5(\eta)$ is chosen so that the contribution to the integral makes it zero. Another important point is that the double infinite series in powers of ξ and $\ln\xi$ introduces an infinity of arbitrary constants, namely, α_{4n+1} if $\alpha_1 \neq 0$ and α_{4n+3} if $\alpha_1 = 0$. This confirms the solution without singularities obtained by Goldstein in § 6 of his paper. It is to be expected that the arbitrary α's will depend on an initial profile.

2. The solution

The solution will follow the method used by Goldstein (1948) and Stewartson (1958).

Take the independent variables

$$\xi = x_1^{\frac{1}{4}}, \quad \eta = \frac{y_1}{2^{\frac{1}{2}}x_1^{\frac{1}{4}}}, \tag{28}$$

and take
$$\psi_1 = 2^{\frac{3}{2}}\xi^3 f(\xi,\eta), \tag{29}$$

where
$$f(\xi,\eta) = \sum_{r=0}^{6} f_r(\eta)\,\xi^r + \xi^5\ln\xi[F_5(\eta)+\xi F_6(\eta)] + O(\xi^7\ln\xi). \tag{30}$$

Then
$$u_1 = 2\xi^2\frac{\partial f}{\partial\eta}, \quad v_1 = \frac{1}{2^{\frac{1}{2}}\xi}\left(3f+\xi\frac{\partial f}{\partial\xi}-\eta\frac{\partial f}{\partial\eta}\right), \tag{31}$$

and by substitution in (19), the equation of motion becomes

$$\frac{\partial^3 f}{\partial\eta^3} - 3f\frac{\partial^2 f}{\partial\eta^2} + 2\left(\frac{\partial f}{\partial\eta}\right)^2 + \xi\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\,\partial\eta} - \frac{\partial^2 f}{\partial\eta^2}\frac{\partial f}{\partial\xi}\right) - \sum_{r=0}^{\infty} P_r\xi^{4r} = 0. \tag{32}$$

The boundary condition (5) becomes

$$\frac{\partial f}{\partial\eta} = 0, \quad \frac{1}{2^{\frac{1}{2}}\xi}\left(3f+\xi\frac{\partial f}{\partial\xi}\right) = -V_s(x_1) \quad \text{at} \quad \eta = 0, \tag{33}$$

where $V_s(x_1)$ is the non-dimensional velocity of suction related to the variables (x_1, y_1). This is connected to the velocity of suction $v_s(x)$ in equation (5) by

$$V_s(x_1) = \left(\frac{U_0}{-(\mathrm{d}U/\mathrm{d}x)\,l}\right)^{\frac{1}{2}} v_s(x). \tag{34}$$

Expanding $V_s(x_1)$ in a power series

$$V_s(x_1) = V_0 + V_1x_1 + V_2x_1^2 + \dots, \tag{35}$$

the second condition in (33) becomes

$$f(\xi,0) = -2^{-\frac{3}{2}}\sum_{r=0}^{\infty}\frac{V_r\xi^{4r+1}}{(r+1)} = -2^{-\frac{3}{2}}\{V_0\xi + \tfrac{1}{2}V_1\xi^5 + \dots\}. \tag{36}$$

The equations and their solutions for $f_0, f_1, \ldots, f_5$ will be considered first.

If ψ_1 is expanded as in equations (29) and (30), then $f_r(\eta)$ will satisfy

$$f_r''' + \sum_{s=0}^{r} [(r-s+2) f_s' f_{r-s}' - (r-s+3) f_s'' f_{r-s}] - P_{\frac{1}{4}r} = 0 \quad (r = 0, 1, \ldots, 5), \tag{37}$$

where $P_{\frac{1}{4}r} = 0$ if $\frac{1}{4}r$ is not an integer.

The equation for $f_5(\eta)$ should also contain the term

$$f_0' F_5' - f_0'' F_5 \tag{38}$$

on the left-hand side. This has been omitted for convenience and it will later be shown that this is zero (see § 5).

The boundary condition (33) becomes

$$\left.\begin{aligned} f_r'(0) &= 0 \quad (r = 0, 1, 2, \ldots, 5), \\ f_r(0) &= 0 \quad (r = 0, 2, 3, 4), \\ f_1(0) &= -2^{-\frac{3}{4}} V_0, \quad f_5(0) = -2^{-\frac{5}{4}} V_1. \end{aligned}\right\} \tag{39}$$

A further boundary condition $\quad f_0''(0) = 0 \quad$ (40)

is introduced. This implies that the series expansion for $\partial u_1/\partial y_1$ starts with a term in $x_1^{\frac{1}{2}}$ as indicated by the numerical work of Hartree (1939*a*), Leigh (1955) and the present author (§§ 14 and 15).

For numerical work later, it is desirable to have the velocity distribution in the neighbourhood of separation. This is

$$u_1 = 2\xi^2 \sum_{r=0}^{r=5} f_r'(\eta)\, \xi^r + F(\xi, \ln \xi), \tag{41}$$

where $F(\xi, \ln \xi)$ contains terms of higher order than ξ^7. This may be written

$$u_1 = \sum_{r=0}^{r=5} a_{r+2} y_1^{r+2}, \tag{42}$$

where $\quad 2^{\frac{1}{2}r} a_{r+2} = \lim_{\eta \to \infty} \dfrac{f_r'}{\eta^{r+2}}, \quad$ (43)

provided that $F(\xi, \ln \xi)$ is negligible. It is easily shown that x_1 will have a very small value before the terms involving $\ln \xi$ become significant and that this value of x_1 is smaller than the accuracy of the numerical results (§ 14.2). Then for the numerical work, provided y_1 is sufficiently small, (42) will give u_1 in the neighbourhood of the separation point.

The equation for $f_0(\eta)$ is $\quad f_0''' - 3 f_0 f_0'' + 2 f_0'^2 = 1, \quad$ (44)

with boundary conditions $\quad f_0(0) = f_0'(0) = f_0''(0) = 0. \quad$ (45)

The solution of (44) satisfying these boundary conditions is

$$f_0(\eta) = \tfrac{1}{6}\eta^3, \tag{46}$$

and, from (43), $\quad a_2 = \tfrac{1}{2}. \quad$ (47)

If we use the solution for $f_0(\eta)$ in equation (46), equation (37) can be written

$$f_r''' - \tfrac{1}{2}\eta^3 f_r'' + \tfrac{1}{2}(r+4)\, \eta^2 f_r' - (r+3)\, \eta f_r = G_r, \tag{48}$$

where $$G_r = \sum_{s=1}^{r-1} [(r-s+3) f_s'' f_{r-s} - (r-s+2) f_s' f_{r-s}'] + P_{\frac{1}{4}r} \quad (r = 1, 2, \ldots, 5) \tag{49}$$

and $P_{\frac{1}{4}r} = 0$ except when $\frac{1}{4}r$ is an integer.

The equation for $f_1(\eta)$ is $$f_1''' - \tfrac{1}{2}\eta^3 f_1'' + \tfrac{5}{2}\eta^2 f_1' - 4\eta f_1 = 0, \tag{50}$$

with boundary conditions $$f_1'(0) = 0, \quad f_1(0) = -2^{-\frac{3}{4}} V_0. \tag{51}$$

The solution of (50) satisfying these boundary conditions is

$$f_1(\eta) = \alpha_1 \eta^2 - 2^{-\frac{3}{4}} V_0 (1 + \tfrac{1}{6}\eta^4), \tag{52}$$

where α_1 is a constant; then, from (43),

$$3!\, a_3 = -V_0. \tag{53}$$

The equation for $f_2(\eta)$ is

$$f_2''' - \tfrac{1}{2}\eta^3 f_2'' + 3\eta^2 f_2' - 5\eta f_2 = -4\alpha_1^2 \eta^2 - 2^{\frac{3}{4}} V_0 \alpha_1 (1 + \tfrac{1}{6}\eta^4) + V_0^2 \eta^2, \tag{54}$$

with boundary conditions $$f_2(0) = f_2'(0) = 0. \tag{55}$$

The solution of equation (54) satisfying these boundary conditions is

$$f_2(\eta) = -\tfrac{1}{15}\alpha_1^2 \eta^5 + \alpha_2 \eta^2 - \tfrac{1}{3}\, 2^{\frac{1}{4}} \alpha_1 V_0 \eta^3 + \tfrac{1}{60} V_0^2 \eta^5, \tag{56}$$

where α_2 is a constant; then, from (43),

$$4!\, a_4 = (V_0^2 - 4\alpha_1^2). \tag{57}$$

Before considering the solution for f_3, the complementary functions of the solution of the equation for $f_r(\eta)$ will be discussed. This has been done by Goldstein (1948, pp. 52–55).

There are three independent complementary functions η^2, g_r and h_r where

$$g_r = -\sum_{m=0}^{\infty} \frac{(m - \frac{3}{2} - \frac{1}{4}r)!\, \frac{1}{4}!\, \eta^{4m+1}}{m!\,(-\frac{3}{2} - \frac{1}{4}r)!\,(m + \frac{1}{4})!\, 8^m (4m-1)} \tag{58}$$

and $$h_r = -\sum_{m=0}^{\infty} \frac{(m - \frac{7}{4} - \frac{1}{4}r)!\,(-\frac{1}{4})!\, \eta^{4m}}{m!\,(-\frac{7}{4} - \frac{1}{4}r)!\,(m - \frac{1}{4})!\, 8^m (2m-1)}. \tag{59}$$

The series for g_r terminates when $r = 4m+2$ and that for h_r terminates when $r = 4m+1$, $m = 0, 1, 2, \ldots$.

Goldstein showed that g_r and h_r are exponentially large as $\eta \to \infty$, except when they are finite series. However, the combination

$$k_r = h_r + \frac{2^{\frac{1}{4}}(-\frac{5}{4})!\,(-\frac{3}{2} - \frac{1}{4}r)!}{(-\frac{3}{4})!\,(-\frac{7}{4} - \frac{1}{4}r)!} g_r \quad (r \neq 4m+1, 4m+2), \tag{60}$$

which is also a complementary function of equation (48), is not exponentially large.

In particular, Goldstein found that

$$k_3 = h_3 - \frac{3 \cdot 2^{\frac{1}{4}} \pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3} g_3 \sim -\frac{\pi}{160(\frac{1}{4}!)^2}\left\{\eta^6 - \frac{15}{2!\,\eta^2} + \frac{15 \cdot 1 \cdot 3}{2 \cdot 3!}\frac{1}{\eta^6} - \frac{15 \cdot 1 \cdot 3 \cdot 3 \cdot 7}{3 \cdot 4!}\frac{1}{\eta^{10}} + \ldots\right\}$$
$$+ \frac{3\pi\eta^2}{32(\frac{1}{4}!)^2}[4 \ln \eta + 2 \ln 2 + \gamma - \tfrac{1}{2}\pi - 5], \tag{61}$$

and $$k_4 = h_4 - \frac{7.2^{\frac{3}{4}}\pi^{\frac{3}{2}}}{64(\frac{1}{4}!)^3} g_4 \sim -\frac{\pi^{\frac{1}{2}}}{48(\frac{1}{4}!)\,2^{\frac{1}{4}}}\left\{\frac{\eta^7}{5} - \frac{3.7}{1!}\eta^3 - \frac{3.1.7.3}{3.2!}\frac{1}{\eta} + \frac{3.1.1.7.3.1}{7.3!}\frac{1}{\eta^5} - \frac{3.1.1.3.7.3.1.5}{11.4!}\frac{1}{\eta^9} + \frac{3.1.1.3.5.7.3.1.5.9}{15.5!}\frac{1}{\eta^{13}} + \dots\right\} - \frac{21.2^{\frac{1}{2}}\pi^{\frac{5}{2}}}{640(\frac{1}{4}!)^4}\eta^2. \tag{62}$$

The equation for f_3 is

$$f_3''' - \tfrac{1}{2}\eta^3 f_3'' + \tfrac{7}{2}\eta^2 f_3' - 6\eta f_3 = -10\alpha_1\alpha_2\eta^2 - \tfrac{4}{3}\alpha_1^3\eta^5 + V_0 2^{\frac{1}{2}}\{-2\alpha_2 + 4\alpha_1^2\eta^3 - \tfrac{1}{2}\alpha_2\eta^4\} + \alpha_1 V_0^2\{4\eta + \tfrac{1}{3}\eta^5\} - \tfrac{1}{3}V_0^3 2^{\frac{1}{2}}\eta^3, \tag{63}$$

with boundary conditions $$f_3(0) = f_3'(0) = 0. \tag{64}$$

The general solution of the differential equation (63) satisfying the boundary conditions (64) is

$$f_3(\eta) = \alpha_3\eta^2 + 4\alpha_1\alpha_2(\eta - g_3) - \tfrac{8}{3}\alpha_1^3(1 + \tfrac{1}{4}\eta^4 - h_3) + \tfrac{1}{3}V_0 2^{\frac{1}{2}}(\tfrac{1}{10}\alpha_1^2\eta^6 - \alpha_2\eta^3) + \tfrac{1}{6}\alpha_1 V_0^2\eta^4 - (2^{\frac{3}{2}}/6!)\,V_0^3\eta^6, \tag{65}$$

where α_3 is a constant.

Since $f_3(\eta)$ must not be exponentially large, it is necessary that g_3 and h_3 appear in the combination k_3. Thus it follows that

$$\alpha_2 = \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}\alpha_1^2}{5(\frac{1}{4}!)^3} \tag{66}$$

and $$f_3(\eta) = \alpha_3\eta^2 + 4\alpha_1\alpha_2\eta - \tfrac{8}{3}\alpha_1^3(1 + \tfrac{1}{4}\eta^4) + \tfrac{8}{3}\alpha_1^3 k_3 + \tfrac{1}{3}V_0 2^{\frac{1}{2}}(\tfrac{1}{10}\alpha_1^2\eta^6 - \alpha_2\eta^3) + \tfrac{1}{6}\alpha_1 V_0^2\eta^4 - (2^{\frac{3}{2}}/6!)\,V_0^3\eta^6. \tag{67}$$

From equation (43), it follows that

$$a_5 = -\frac{2^{\frac{1}{2}}\pi\alpha_1^3}{40(\frac{1}{4}!)^2} - \frac{V_0}{5!}(V_0^2 - 12\alpha_1^2). \tag{68}$$

The function $k_3(\eta)$ is a complementary function of $f_3(\eta)$ so that $k_3(\eta)$ must satisfy

$$k_3''' - \tfrac{1}{2}\eta^3 k_3'' + \tfrac{7}{2}\eta^2 k_3' - 6\eta k_3 = 0. \tag{69}$$

The differential equation for $f_4(\eta)$ is obtained by substituting for f_1, f_2 and f_3 and their derivatives from equations (52), (56) and (67) into equations (48) and (49). Then $f_4(\eta)$ satisfies

$$f_4''' - \tfrac{1}{2}\eta^3 f_4'' + 4\eta^2 f_4' - 7\eta f_4 = -2\alpha_1^2\alpha_2\eta^5 + \tfrac{8}{3}\alpha_1^4\eta^4 - 6(\alpha_2^2 + 2\alpha_1\alpha_3)\,\eta^2 - 16\alpha_1^2\alpha_2\eta - 32\alpha_1^4 + P_1 + \tfrac{32}{3}\alpha_1^4(\eta^2 k_3'' - 4\eta k_3' + 3k_3) + V_0 2^{\frac{1}{2}}\{\tfrac{6}{5}\alpha_1^3\eta^6 - \tfrac{2}{3}\alpha_3\eta^4 + 16\alpha_1^3\eta^2 - 2\alpha_3 - \tfrac{8}{3}\alpha_1^3[(1 + \tfrac{1}{6}\eta^4)\,k_3'' - \tfrac{4}{3}\eta^3 k_3' + 3\eta^2 k_3]\} + V_0^2\{\tfrac{1}{2}\alpha_2\eta^5 - 4\alpha_1^2\eta^4 + 4\alpha_2\eta\} + V_0^3 2^{\frac{1}{2}}\{-\tfrac{1}{10}\alpha_1\eta^6 - 2\alpha_1\eta^2\} + \tfrac{1}{6}V_0^4\eta^4, \tag{70}$$

with boundary conditions $$f_4(0) = f_4'(0) = 0. \tag{71}$$

The complementary functions of $f_4(\eta)$ have already been discussed and a particular integral is easily obtained for the polynomial part of the right-hand side of (70). Thus only a particular integral of

$$f''' - \tfrac{1}{2}\eta^3 f'' + 4\eta^2 f' - 7\eta f = \tfrac{32}{3}\alpha_1^4(\eta^2 k_3'' - 4\eta k_3' + 3k_3) - \tfrac{8}{3}V_0 2^{\frac{1}{2}}\alpha_1^3[(1 + \tfrac{1}{6}\eta^4)\,k_3'' - \tfrac{4}{3}\eta^3 k_3' + 3\eta^2 k_3] \tag{72}$$

is required. This was obtained by looking for a solution

$$f(\eta) = P(\eta)\,k_3'' + Q(\eta)\,k_3' + R(\eta)\,k_3, \tag{73}$$

where $P(\eta)$, $Q(\eta)$ and $R(\eta)$ are polynomials and by using relation (69) for k_3'''. In this way the general solution of the differential equation (70) satisfying the boundary conditions (71) was found to be

$$\begin{aligned} f_4(\eta) = {} & \alpha_4\eta^2+\tfrac{1}{6}P_1(\eta^3-\tfrac{1}{105}\eta^7)+2(\alpha_2^2+2\alpha_1\alpha_3)(\eta-g_4)-\tfrac{2}{3}\alpha_1^2\alpha_2\eta^4 \\ & +\tfrac{16}{3}\alpha_1^4\left(k_3'+\frac{3.2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3}h_4-\eta^3+\tfrac{1}{84}\eta^7\right) \\ & +V_0 2^{\frac{1}{2}}\{\tfrac{8}{3}\alpha_1^3(\eta-g_4)-\tfrac{1}{3}\alpha_3\eta^3+\tfrac{2}{5}\alpha_1^3\eta^5-\tfrac{4}{27}\alpha_1^3[4(\eta k_3-g_4)+\eta^2k_3']\} \\ & +V_0^2(\tfrac{1}{6}\alpha_2\eta^4-\tfrac{2}{105}\alpha_1^2\eta^7)-\tfrac{1}{30}V_0^3 2^{\frac{1}{2}}\alpha_1\eta^5+\tfrac{1}{1260}V_0^4\eta^7, \end{aligned} \tag{74}$$

where α_4 is a constant.

The only exponentially large terms in equation (74) are g_4 and h_4. Since $f_4(\eta)$ must not be exponentially large, the terms g_4 and h_4 must appear in (74) in the combination $k_4(\eta)$. The terms containing g_4 and h_4 are

$$\frac{8.2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^4h_4-2(\alpha_2^2+2\alpha_1\alpha_3)g_4-\tfrac{56}{27}\alpha_1^3V_0 2^{\frac{1}{2}}g_4. \tag{75}$$

Comparison with (62) yields

$$2(\alpha_2^2+2\alpha_1\alpha_3)+\tfrac{56}{27}\alpha_1^3V_0 2^{\frac{1}{2}}=\frac{7\pi^3}{20(\frac{1}{4}!)^6}\alpha_1^4, \tag{76}$$

and substituting the value of α_2 from (66) gives

$$\alpha_3=\frac{\pi^3}{400(\frac{1}{4}!)^6}(35-8.2^{\frac{1}{2}})\alpha_1^3-\tfrac{14}{27}\alpha_1^2V_0 2^{\frac{1}{2}}. \tag{77}$$

It follows that

$$\begin{aligned} f_4(\eta) = {} & \alpha_4\eta^2+\tfrac{1}{6}P_1(\eta^3-\tfrac{1}{105}\eta^7)+2(\alpha_2^2+2\alpha_1\alpha_3)\eta-\tfrac{2}{3}\alpha_1^2\alpha_2\eta^4+8\alpha_1^2\alpha_2k_4+\tfrac{16}{3}\alpha_1^4(k_3'-\eta^3+\tfrac{1}{84}\eta^7) \\ & +V_0 2^{\frac{1}{2}}\{\tfrac{8}{3}\alpha_1^3\eta-\tfrac{1}{3}\alpha_3\eta^3+\tfrac{2}{5}\alpha_1^3\eta^5-\tfrac{4}{27}\alpha_1^3[4\eta k_3+\eta^2k_3']\} \\ & +V_0^2(\tfrac{1}{6}\alpha_2\eta^4-\tfrac{2}{105}\alpha_1^2\eta^7)-\tfrac{1}{30}V_0^3 2^{\frac{1}{2}}\alpha_1\eta^5+\tfrac{1}{1260}V_0^4\eta^7. \end{aligned} \tag{78}$$

Then

$$f_4''(0)=\alpha_4+\tfrac{10}{9}\alpha_1\alpha_2V_0 2^{\frac{1}{2}} \tag{79}$$

so that when $V_0 \neq 0$, $f_4''(0)$ is not equal to the constant of integration α_4 as in the case of no suction. For convenience α_r will be taken as the constant of integration and not as $f_r''(0)$.

When $V_0=0$ the solution (78) is equivalent to the form given by Goldstein.

It should be noted that $f_3(\eta)$ in equation (67) and $f_4(\eta)$ in equation (78) are not given explicitly in increasing powers of V_0 since they contain the constants α_3 and α_4 which are functions of V_0.

It follows from (43) that

$$a_6=\left(\frac{1}{9}-\frac{7\pi^2}{600(\frac{1}{4}!)^4}\right)\alpha_1^4-\tfrac{1}{360}P_1+\alpha_1^3V_0 2^{\frac{1}{2}}\left(\frac{7\pi}{432(\frac{1}{4}!)^2}\right)-\tfrac{1}{30}\alpha_1^2V_0^2+\tfrac{1}{720}V_0^4. \tag{80}$$

The function $k_4(\eta)$ is a complementary function of $f_4(\eta)$ so that $k_4(\eta)$ must satisfy

$$k_4'''-\tfrac{1}{2}\eta^3k_4''+4\eta^2k_4'-7\eta k_4=0. \tag{81}$$

The differential equation for $f_5(\eta)$ is obtained by substituting for f_1, f_2, f_3 and f_4 and their derivatives from equations (52), (56), (67) and (78) into equations (48) and (49). Then, by using equations (69) and (81) for the replacement of k_3''' and k_4''' respectively, $f_5(\eta)$ is found to satisfy

$$\begin{aligned} f_5''' - \tfrac{1}{2}\eta^3 f_5'' + \tfrac{9}{2}\eta^2 f_5' - 8\eta f_5 \\ = 16\alpha_1^3\alpha_2(2\eta^2 k_4'' - 9\eta k_4' + 7k_4) + \tfrac{8}{3}\alpha_1^3\{(\tfrac{11}{3}\alpha_1^2\eta^5 - 36\alpha_1^2\eta + 5\alpha_2\eta^2)\,k_3'' - (25\alpha_1^2\eta^4 - 28\alpha_1^2 + 18\alpha_2\eta)\,k_3' \\ + (40\alpha_1^2\eta^3 + 12\alpha_2)\,k_3\} + (-\tfrac{4}{45}\alpha_1 P_1 + \tfrac{32}{9}\alpha_1^5)\,\eta^7 - \tfrac{8}{3}\alpha_1^2\alpha_3\eta^5 - \tfrac{40}{3}\alpha_1^3\alpha_2\eta^4 + \tfrac{8}{3}\alpha_1(-P_1 + 40\alpha_1^4)\,\eta^3 \\ - 14(\alpha_2\alpha_3 + \alpha_1\alpha_4)\,\eta^2 - 16(2\alpha_2^2 + \alpha_1\alpha_3)\,\alpha_1\eta - 32\alpha_1^3\alpha_2 + V_0 2^{\frac{1}{2}}\{-8\alpha_1^2\alpha_2[(1 + \tfrac{1}{6}\eta^4)\,k_4'' \\ - \tfrac{3}{2}\eta^3 k_4' + \tfrac{7}{2}\eta^2 k_4] + \tfrac{8}{3}\alpha_1^4[-(\tfrac{5}{18}\eta^7 + \tfrac{4}{9}\eta^3)\,k_3'' + (\tfrac{35}{18}\eta^6 + 12\eta^2)\,k_3' - (\tfrac{10}{3}\eta^5 + \tfrac{208}{9}\eta)\,k_3] \\ - \alpha_4(\tfrac{5}{6}\eta^4 + 2) + P_1(\tfrac{1}{15}\eta^5 - \eta) + \alpha_1^2\alpha_2(\tfrac{9}{5}\eta^6 - 4\eta^2) - \alpha_1^4(\tfrac{56}{15}\eta^5 - \tfrac{160}{3}\eta)\} \\ + V_0^2\{\tfrac{8}{27}\alpha_1^3[(\tfrac{1}{12}\eta^9 + \tfrac{13}{12}\eta^5 + 8\eta)\,k_3'' + (-\tfrac{7}{12}\eta^8 - \tfrac{169}{12}\eta^4 + 10)k_3' + (\eta^7 + 32\eta^3)\,k_3] + \alpha_3(\tfrac{2}{3}\eta^5 + 4\eta) \\ - \alpha_1^3(\tfrac{16}{15}\eta^7 + 32\eta^3)\} + V_0^3 2^{\frac{1}{2}}\{\tfrac{4}{3}\alpha_1^2\eta^5 - \alpha_2(\tfrac{3}{20}\eta^6 + 2\eta^2)\} + V_0^4\alpha_1(\tfrac{2}{45}\eta^7 + \tfrac{4}{3}\eta^3) + V_0^5 2^{\frac{1}{2}}(-\tfrac{1}{30}\eta^5), \end{aligned} \tag{82}$$

with boundary conditions $$f_5(0) = -2^{-\frac{5}{2}}V_1, \quad f_5'(0) = 0. \tag{83}$$

The complementary functions for $f_5(\eta)$ are η^2, g_5 and h_5 where h_5 is the terminating series

$$h_5 = 1 + \tfrac{1}{3}\eta^4 - \tfrac{1}{252}\eta^8. \tag{84}$$

Thus g_5 is the only complementary function containing exponentially large terms. A particular integral is easily obtained for the polynomial part of the right-hand side of (82). A particular integral for the terms involving k_4 is obtained by looking for a solution of the form

$$f(\eta) = P(\eta)\,k_4'' + Q(\eta)\,k_4' + R(\eta)k_4, \tag{85}$$

where $P(\eta)$, $Q(\eta)$ and $R(\eta)$ are polynomials, and by using relation (81) for k_4'''. This also yields a simple solution.

However, a particular integral for the terms involving k_3 is not as easy. If a solution of the form (73) is tried, it is necessary for $P(\eta)$, $Q(\eta)$ or $R(\eta)$ to be an infinite series. Because of this, a simple solution is chosen which will satisfy the differential equation for most of the k_3 terms on the right-hand side of equation (82) and the one selected was such that only a multiple of k_3' remained on the right-hand side. Hence the general solution of (82) can be obtained if a particular integral of

$$f''' - \tfrac{1}{2}\eta^3 f'' + \tfrac{9}{2}\eta^2 f' - 8\eta f = k_3' \tag{86}$$

is found.

Using equations (61), (58) and (59) to express k_3' as a power series, we can write equation (86) as

$$f''' - \tfrac{1}{2}\eta^3 f'' + \tfrac{9}{2}\eta^2 f' - 8\eta f = \sum_{n=1}^{\infty} 4na_n\eta^{4n-1} - C\sum_{n=0}^{\infty}(4n+1)\,b_n\eta^{4n}, \tag{87}$$

where

$$\left.\begin{aligned} a_n &= -\frac{(n-\frac{5}{2})!\,(-\frac{1}{4})!}{n!\,(-\frac{5}{2})!\,(n-\frac{1}{4})!\,8^n(2n-1)}, \\ b_n &= -\frac{(n-\frac{9}{4})!\,\frac{1}{4}!}{n!\,(-\frac{9}{4})!\,(n+\frac{1}{4})!\,8^n(4n-1)}, \end{aligned}\right\} \tag{88}$$

and

$$C = \frac{3\,.\,2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3}. \tag{89}$$

This has a series solution $$f = \sum_{p=1}^{\infty}\alpha_p\eta^{4p+2} - C\sum_{p=0}^{\infty}\beta_p\eta^{4p+3}, \tag{90}$$

where $$\alpha_p = -\frac{\frac{1}{4}!}{32(p+\frac{1}{2})\,p(p-\frac{1}{2})\,(p-\frac{3}{2})\,(p+\frac{1}{4})!\,8^p}\sum_{k=0}^{p-1}\frac{(k-\frac{1}{2})!\,(k+\frac{1}{4})!\,\frac{3}{4}!}{(-\frac{1}{2})!\,\frac{1}{4}!\,(k+\frac{3}{4})!\,k!},\tag{91}$$

and $$\beta_p = -\frac{1}{64(p+\frac{3}{4})\,(p+\frac{1}{4})\,(p-\frac{1}{4})\,(p-\frac{5}{4})\,(p+\frac{1}{2})!\,8^p}\frac{\frac{1}{4}!}{(-\frac{9}{4})!}\sum_{n=0}^{p}\frac{(n-\frac{5}{4})!\,(n-\frac{1}{2})!}{n!\,(n-\frac{3}{4})!}.\tag{92}$$

Let $F(p)$ be an analytic function of p such that

$$F(p+1)-F(p) = \frac{(p-\frac{1}{2})!\,(p+\frac{1}{4})!\,\frac{3}{4}!}{(-\frac{1}{2})!\,\frac{1}{4}!\,(p+\frac{3}{4})!\,p!},\tag{93}$$

and $$F(0) = 0.\tag{94}$$

Then, in terms of $F(p)$, α_p and β_p can be written

$$\alpha_p = -\frac{(\frac{1}{4}!)\,F(p)}{32(p+\frac{1}{2})\,p(p-\frac{1}{2})\,(p-\frac{3}{2})\,(p+\frac{1}{4})!\,8^p},\tag{95}$$

$$\beta_p = -\frac{10(\frac{1}{4}!)^4}{3\pi^{\frac{3}{2}}}\,\frac{\{F(p+\frac{1}{4})-F(-\frac{3}{4})\}}{64(p+\frac{3}{4})\,(p+\frac{1}{4})\,(p-\frac{1}{4})\,(p-\frac{5}{4})\,(p+\frac{1}{2})!\,8^p}.\tag{96}$$

From equations (93) and (94)

$$F(p) = \left\{{}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}\\ & \frac{7}{4}\end{matrix}\right]\right\}_{p\ \text{terms}} \quad\text{when}\quad p = 0, 1, 2, \ldots.\tag{97}$$

All the hypergeometric functions that are considered have unit argument. Bailey (1935, p. 93) has given the sum of the hypergeometric series ${}_2F_1\left[\begin{matrix}a, & b\\ & f\end{matrix}\right]$ to p terms provided $f \geqslant a+b$ and p is a positive integer. From this,

$$F(p) = \frac{(p-\frac{1}{2})!\,(p+\frac{1}{4})!}{(p-1)!\,(p+\frac{3}{4})!}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & p+\frac{3}{4}\\ & \frac{7}{4}, & p+\frac{7}{4}\end{matrix}\right].\tag{98}$$

Bailey's result is proved only for p a positive integer but the same formulae lead to the result (93) for non-integral p.

In particular $$F(-\tfrac{3}{4}) = \frac{3\pi^{\frac{3}{2}}2^{\frac{1}{2}}}{16(\frac{1}{4}!)^2}.\tag{99}$$

Hence the series solution for $f(\eta)$ is

$$\begin{aligned} f(\eta) = \frac{(\frac{1}{4}!)}{32}\Bigg\{&\sum_{p=0}^{\infty}\frac{F(p+\frac{1}{4})\,\eta^{4p+3}}{(p+\frac{3}{4})\,(p+\frac{1}{4})\,(p-\frac{1}{4})\,(p-\frac{5}{4})\,(p+\frac{1}{2})!\,8^{p+\frac{1}{4}}} \\ &-\sum_{p=1}^{\infty}\frac{F(p)\,\eta^{4p+2}}{(p+\frac{1}{2})\,p(p-\frac{1}{2})\,(p-\frac{3}{2})\,(p+\frac{1}{4})!\,8^p}\Bigg\} \\ &-\sum_{p=0}^{\infty}\frac{3\pi^{\frac{3}{2}}2^{\frac{3}{4}}\eta^{4p+3}}{4^5(\frac{1}{4}!)\,(p+\frac{3}{4})\,(p+\frac{1}{4})\,(p-\frac{1}{4})\,(p-\frac{5}{4})\,(p+\frac{1}{2})!\,8^p}.\end{aligned}\tag{100}$$

Since there is only one exponentially large complementary function g_5, it is expected that f_5 will behave asymptotically like a multiple of g_5. From equation (58), g_5 can be written

$$g_5 = -\frac{21}{4^4}\sum_{m=0}^{\infty}\frac{\eta^{4m+1}}{(m+\frac{1}{4})\,(m-\frac{1}{4})\,(m-\frac{3}{4})\,(m-\frac{7}{4})\,m!\,8^m}.\tag{101}$$

Consider the contour integral

$$\frac{1}{2\pi \mathrm{i}}\int \frac{2\pi\eta^{4s+1}\,\mathrm{d}s}{\sin(2\pi s)\,(s+\frac{1}{4})\,(s-\frac{1}{4})\,(s-\frac{3}{4})\,(s-\frac{7}{4})\,s!\,8^s}. \tag{102}$$

The path of integration is taken around a contour consisting of the straight line from $-N-\infty\,\mathrm{i}$ to $-N+\infty\,\mathrm{i}$ and the part of a circle of infinite radius to the right of that line. In this way it may be shown that

$$-\sum_{n=0}^{\infty}\frac{3\pi^{\frac{3}{2}}2^{\frac{3}{4}}\eta^{4n+3}}{4^5(\frac{1}{4}!)\,(n+\frac{3}{4})\,(n+\frac{1}{4})\,(n-\frac{1}{4})\,(n-\frac{5}{4})\,(n+\frac{1}{2})!\,8^n}-\frac{\pi^{\frac{3}{2}}2^{\frac{1}{4}}}{7(\frac{1}{4}!)}g_5$$
$$\sim\frac{3\pi^{\frac{1}{2}}2^{\frac{7}{4}}}{(\frac{1}{4}!)}\sum_{n=0}^{\infty}\frac{(-8)^n\,(n-\frac{1}{2})!}{(4n+1)\,(4n+3)\,(4n+5)\,(4n+9)\,\eta^{4n+1}}-\frac{3\pi^{\frac{3}{2}}2^{\frac{1}{2}}h_5}{32}-\frac{2^{\frac{1}{2}}\pi^{\frac{5}{2}}\eta^2}{32(\frac{1}{4}!)^2}. \tag{103}$$

Thus equation (103) gives the asymptotic expansion of the third term in equation (100). To discuss the first two terms of equation (100) write them as

$$\frac{(\frac{1}{4}!)}{32}\left\{\sum_{p=0}^{\infty}\frac{(p-\frac{9}{4})!\,{}_3F_2\begin{bmatrix}\frac{1}{2},\ \frac{5}{4},\ p+1\\ \frac{7}{4},\ p+2\end{bmatrix}\eta^{4p+3}}{(p+\frac{1}{4})!\,(p+1)!\,(p+\frac{3}{4})\,8^{p+\frac{1}{4}}}-\sum_{p=1}^{\infty}\frac{(p-\frac{5}{2})!\,{}_3F_2\begin{bmatrix}\frac{1}{2},\ \frac{5}{4},\ p+\frac{3}{4}\\ \frac{7}{4},\ p+\frac{7}{4}\end{bmatrix}\eta^{4p+2}}{p!\,(p+\frac{3}{4})!\,(p+\frac{1}{2})\,8^p}\right\} \tag{104}$$

and consider the contour integral

$$\frac{1}{2\pi\mathrm{i}}\int\frac{(-s-1)!\,(-s-\frac{7}{4})!\,(s-\frac{5}{2})!\,{}_3F_2\begin{bmatrix}\frac{1}{2},\ \frac{5}{4},\ s+\frac{3}{4}\\ \frac{7}{4},\ s+\frac{7}{4}\end{bmatrix}\eta^{4s+2}}{(s+\frac{1}{2})\,8^s}ds \tag{105}$$

taken around the same contour as before. In this way it is found that

$$\pi 2^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(n-\frac{5}{2})!\,{}_3F_2\begin{bmatrix}\frac{1}{2},\ \frac{5}{4},\ n+\frac{3}{4}\\ \frac{7}{4},\ n+\frac{7}{4}\end{bmatrix}\eta^{4n+2}}{n!\,(n+\frac{3}{4})!\,(n+\frac{1}{2})\,8^n}-\sum_{n=0}^{\infty}\frac{(n-\frac{13}{4})!\,{}_3F_2\begin{bmatrix}\frac{1}{2},\ \frac{5}{4},\ n\\ \frac{7}{4},\ n+1\end{bmatrix}\eta^{4n-1}}{n!\,(n-\frac{3}{4})!\,(n-\frac{1}{4})\,8^{n-\frac{3}{4}}}\right\}$$
$$\sim\frac{(-\frac{1}{2})!\,(-\frac{5}{4})!}{2.8^{-\frac{1}{2}}}\Big\{\phi(2)\,[-\mathscr{F}(-\tfrac{1}{2})-\mathscr{F}(-\tfrac{5}{4})+\mathscr{F}(0)+\tfrac{3}{2}+4\ln\eta-\ln 8]$$
$$+\left(\frac{\mathrm{d}}{\mathrm{d}s}{}_3F_2\begin{bmatrix}\frac{1}{2},\ \frac{5}{4},\ s+\frac{3}{4}\\ \frac{7}{4},\ s+\frac{7}{4}\end{bmatrix}\right)_{s=-\frac{1}{2}}\Big\}+\sum_{\substack{n=0\\ n\neq 2}}^{\infty}\frac{(-1)^n\,(n-\frac{5}{2})!\,(n-\frac{13}{4})!\,\phi(n)\,\eta^{-4n+8}}{n!\,(2-n)\,8^{-n+\frac{3}{2}}}$$
$$-\frac{3\pi^{\frac{3}{2}}}{2^{\frac{3}{4}}(\frac{1}{4}!)^2}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})!\,(-8)^{n+1}}{(n+\frac{13}{4})\,(n+\frac{9}{4})\,(n+\frac{7}{4})\,(n+\frac{5}{4})\,\eta^{4n+5}}, \tag{106}$$

where
$$\mathscr{F}(s)=\frac{\mathrm{d}}{\mathrm{d}s}\ln s! \tag{107}$$

and
$$\phi(n)={}_3F_2\begin{bmatrix}\frac{1}{2},\ \frac{5}{4},\ -n+\frac{9}{4}\\ \frac{7}{4},\ -n+\frac{13}{4}\end{bmatrix}. \tag{108}$$

It can be shown that $\phi(n)$ satisfies

$$\frac{(n-1)\,(n-\frac{7}{4})}{(n-\frac{1}{2})\,(n-\frac{9}{4})}\phi(n)-\phi(n+1)=\frac{(\frac{3}{4}!)}{(-\frac{1}{2})!\,(\frac{1}{4}!)\,(\frac{1}{2}-n)}. \tag{109}$$

In particular,
$$\phi(2)=\frac{2(\frac{3}{4}!)}{(-\frac{1}{2})!\,(\frac{1}{4}!)}, \tag{110}$$

so that the values of $\phi(3), \phi(4), \ldots$ can be deduced in succession. If

$$\Phi(n) = \frac{(n-\frac{3}{2})!\,(n-\frac{13}{4})!}{(n-2)!\,(n-\frac{11}{4})!}\phi(n), \quad \Phi(1) = 0, \tag{111}$$

it follows that

$$\Phi(n) = -\frac{3\pi^2}{32(\frac{1}{4}!)^4}\left\{{}_2F_1\left[\begin{matrix}\frac{1}{2}, & -\frac{1}{4} \\ & \frac{1}{4}\end{matrix}\right]\right\}_{n-1\text{ terms}}. \tag{112}$$

Also it can be shown that

$$\left(\frac{d}{ds}{}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & s+\frac{3}{4} \\ & \frac{7}{4}, & s+\frac{7}{4}\end{matrix}\right]\right)_{s=-\frac{1}{2}} = \tfrac{8}{35}\,{}_3F_2\left[\begin{matrix}\frac{3}{2}, & \frac{5}{4}, & \frac{5}{4} \\ & \frac{11}{4}, & \frac{9}{4}\end{matrix}\right] \tag{113}$$

$$= \frac{6\pi^{\frac{1}{2}}\,2^{\frac{1}{2}}}{5(\frac{1}{4}!)^2} - 4\pi^{-\frac{1}{2}}\,2^{\frac{1}{2}}(\tfrac{1}{4}!)^2 F(\tfrac{3}{2}). \tag{114}$$

Finally, with the use of

$$\left.\begin{aligned}\phi(0) &= \frac{(\frac{9}{4}!)\,(\frac{1}{2}!)}{(\frac{7}{4}!)}F(\tfrac{3}{2}), \\ \phi(1) &= \frac{14(\frac{1}{2}!)\,(\frac{9}{4}!)}{9(\frac{7}{4}!)}F(\tfrac{3}{2}) - \frac{(\frac{3}{4}!)}{(\frac{1}{2}!)\,(\frac{1}{4}!)},\end{aligned}\right\} \tag{115}$$

and

$$\left.\begin{aligned}\mathscr{F}(-\tfrac{5}{4}) &= \tfrac{1}{2}\pi + 4 - 3\ln 2 + \gamma, \\ \mathscr{F}(0) &= -\gamma, \\ \mathscr{F}(-\tfrac{1}{2}) &= -\gamma - 2\ln 2,\end{aligned}\right\} \tag{116}$$

where γ is Euler's constant, the asymptotic expansion of the first two terms of f in equation (100) can be shown to be

$$\begin{aligned}&\sim \frac{3\pi}{512(\frac{1}{4}!)^3}\sum_{n=1}^{\infty}\frac{(n-\frac{3}{4})!\,(-8)^n}{(n+2)\,(n+1)\,(n+\frac{1}{2})\,n\eta^{4n}}\left\{{}_2F_1\left[\begin{matrix}\frac{1}{2}, & -\frac{1}{4} \\ & \frac{1}{4}\end{matrix}\right]\right\}_{n+1\text{ terms}} \\ &\quad - \frac{3\pi^{\frac{1}{2}}}{64.2^{\frac{1}{4}}(\frac{1}{4}!)}\sum_{n=0}^{\infty}\frac{(n-\frac{1}{2})!\,(-8)^n}{(n+\frac{9}{4})\,(n+\frac{5}{4})\,(n+\frac{3}{4})\,(n+\frac{1}{4})\,\eta^{4n+1}} + \frac{2^{\frac{3}{2}}(\frac{1}{4}!)^2}{9\pi^{\frac{1}{2}}}\phi(\tfrac{3}{2})\,\eta^2 + \tfrac{1}{4}(\tfrac{1}{4}!)^2 F(\tfrac{3}{2})\,h_5(\eta) \\ &\quad - \frac{3\pi}{40(\frac{1}{4}!)^2}\{1+\tfrac{1}{8}\eta^4\} - \frac{3\pi}{128(\frac{1}{4}!)^2}\{\gamma + 2\ln 2 - \tfrac{1}{2}\pi - \tfrac{5}{2} + 4\ln\eta\}.\end{aligned} \tag{117}$$

An explicit formula for $F(\frac{3}{2})$ has not yet been found, but it can be computed, since $F(m+\frac{1}{2})$ for, say, $m = 5$ can be found by interpolation from the values of $F(p)$ for integral p and thus $F(\frac{3}{2})$ will follow by use of the recurrence relation (93). A check is possible since the values of $F(p+\frac{1}{4})$ are also known.

Again, it does not seem possible to evaluate $\phi(\frac{3}{2})$ but a similar method is applicable. $\Phi(n+\frac{1}{2})$ may be calculated by interpolating between integral values of $\Phi(n)$ given by (112) and hence $\phi(\frac{3}{2})$ from (111) and recurrence relation (109).

On combining (117) with (103), the asymptotic expansion for f is

$$\begin{aligned}f(\eta) - \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{7(\frac{1}{4}!)}g_5 &\sim \{\tfrac{1}{4}(\tfrac{1}{4}!)^2 F(\tfrac{3}{2}) - \tfrac{3}{32}\pi^{\frac{3}{2}}\,2^{\frac{1}{2}}\}h_5 - \frac{3\pi\eta^4}{320(\frac{1}{4}!)^2} \\ &\quad + \left\{\frac{2^{\frac{3}{2}}(\frac{1}{4}!)^2}{9\pi^{\frac{1}{2}}}\phi(\tfrac{3}{2}) - \frac{2^{\frac{1}{2}}\pi^{\frac{5}{2}}}{32(\frac{1}{4}!)^2}\right\}\eta^2 - \frac{3\pi}{128(\frac{1}{4}!)^2}\{\gamma + 2\ln 2 - \tfrac{1}{2}\pi + \tfrac{7}{10} + 4\ln\eta\} \\ &\quad + \frac{3\pi}{512(\frac{1}{4}!)^3}\sum_{n=1}^{\infty}\frac{(-8)^n\,(n-\frac{3}{4})!}{(n+2)\,(n+1)\,(n+\frac{1}{2})\,n\eta^{4n}}\left\{{}_2F_1\left[\begin{matrix}\frac{1}{2}, & -\frac{1}{4} \\ & \frac{1}{4}\end{matrix}\right]\right\}_{n+1\text{ terms}}.\end{aligned} \tag{118}$$

The solution $f(\eta)$ of equation (86) will be written

$$f(\eta) = p_5(\eta) + \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{7(\frac{1}{4}!)} g_5(\eta), \tag{119}$$

so that $p_5(\eta)$ is not exponentially large.

Hence the general solution of equation (82) is

$$\begin{aligned} f_5(\eta) = {} & \alpha_5\eta^2 + \beta_1 h_5 + \beta_2 g_5 + \tfrac{1}{45}\alpha_1(-P_1 + 40\alpha_1^4)\,\eta^6 - \tfrac{2}{3}\alpha_1^2\alpha_3\eta^4 - \tfrac{16}{3}\alpha_1^3\alpha_2\eta^3 + 4(\alpha_1\alpha_4 + \alpha_2\alpha_3)\,\eta \\ & + 4\alpha_1\alpha_2^2 + 16\alpha_1^3\alpha_2(\tfrac{1}{3}k_3' + k_4') + \alpha_1^5(-\tfrac{5}{12}\eta^2 k_3 - \tfrac{49}{72}\eta^3 k_3' + \tfrac{25}{4}k_3'') + 50\alpha_1^5\left\{p_5(\eta) + \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{7(\frac{1}{4}!)} g_5\right\} \\ & + V_0 2^{\frac{1}{2}}\{-\tfrac{4}{9}\alpha_1^2\alpha_2(4\eta k_4 + \eta^2 k_4') - \tfrac{8}{27}\alpha_1^4(4k_3 + 6\eta k_3' + \eta^2 k_3'') + \alpha_1^2\alpha_2(\tfrac{2}{5}\eta^5 + 8\eta) + \tfrac{1}{120}P_1(2\eta^4 + 21) \\ & - \tfrac{2}{15}\alpha^4(7\eta^4 + 71) - \tfrac{1}{3}\alpha_4\eta^3\} + V_0^2\left\{\tfrac{1}{1944}\alpha_1^3[(16\eta^4 + 117)\,k_3'' + \tfrac{251}{2}\eta^3 k_3' + 581\eta^2 k_3] + \tfrac{1}{6}\alpha_3\eta^4\right. \\ & \left. - \tfrac{4}{15}\alpha_1^3\eta^6 + \tfrac{13}{27}\alpha_1^3\left[p_5(\eta) + \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{7(\frac{1}{4}!)} g_5\right]\right\} \\ & + V_0^3 2^{\frac{1}{2}}\{-\tfrac{1}{30}\alpha_2\eta^5 + \alpha_1^2(\tfrac{1}{3}\eta^4 + 1)\} + \tfrac{1}{90}\alpha_1 V_0^4\eta^6 - \tfrac{1}{120}V_0^5 2^{\frac{1}{2}}(3 + \eta^4), \end{aligned} \tag{120}$$

where α_5, β_1 and β_2 are constants. The first boundary condition in (83) gives

$$\beta_1 = 8\alpha_1(\alpha_2^2 + \alpha_1\alpha_3) + V_0 2^{\frac{1}{2}}(\tfrac{74}{5}\alpha_1^4 - \tfrac{7}{40}P_1) - \alpha_1^2 V_0^3 2^{\frac{1}{2}} + \tfrac{1}{40}V_0^5 2^{\frac{1}{2}} - \tfrac{1}{8}V_1 2^{\frac{1}{2}}, \tag{121}$$

and the second boundary condition in (83) gives

$$\beta_2 = -4(\alpha_2\alpha_3 + \alpha_1\alpha_4) - \tfrac{32}{3}\alpha_1^2\alpha_2 V_0 2^{\frac{1}{2}}. \tag{122}$$

In order that $f_5(\eta)$ shall not contain exponentially large terms the coefficient of g_5 must be zero. Thus

$$\beta_2 + \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{7(\frac{1}{4}!)}\{50\alpha_1^5 + \tfrac{13}{27}\alpha_1^3 V_0^2\} = 0. \tag{123}$$

Substituting for β_2 from equation (122) α_4 satisfies

$$\alpha_4 = \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{28(\frac{1}{4}!)}\{50\alpha_1^4 + \tfrac{13}{27}\alpha_1^2 V_0^2\} - \frac{\alpha_2\alpha_3}{\alpha_1} - \tfrac{8}{3}\alpha_1\alpha_2 V_0 2^{\frac{1}{2}}, \tag{124}$$

where α_2 and α_3 are given by equations (66) and (77).

3. The solution for $f_5(\eta)$ without suction

When there is no suction, a check on the solution for $f_5(\eta)$ is possible by comparison with the numerical work of Jones (1948). In this case the solution (120) for f_5, with the conditions (121), (122) and (123) inserted, reduces to

$$\begin{aligned} f_5(\eta) = {} & \alpha_5\eta^2 + 8\alpha_1(\alpha_2^2 + \alpha_1\alpha_3)\,h_5 + \tfrac{1}{45}\alpha_1(-P_1 + 40\alpha_1^4)\,\eta^6 - \tfrac{2}{3}\alpha_1^2\alpha_3\eta^4 - \tfrac{16}{3}\alpha_1^3\alpha_2\eta^3 + 4(\alpha_2\alpha_3 + \alpha_1\alpha_4)\eta \\ & + 4\alpha_1\alpha_2^2 + 16\alpha_1^3\alpha_2(\tfrac{1}{3}k_3' + k_4') + \alpha_1^5\{-\tfrac{5}{12}\eta^2 k_3 - \tfrac{49}{72}\eta^3 k_3' + \tfrac{25}{4}k_3''\} + 50\alpha_1^5 p_5(\eta). \end{aligned} \tag{125}$$

The condition (124) for α_4 gives

$$\alpha_4 = \frac{25\,.\,2^{\frac{1}{4}}\pi^{\frac{3}{2}}\alpha_1^4}{14(\frac{1}{4}!)} - \frac{\alpha_2\alpha_3}{\alpha_1}. \tag{126}$$

The numerical values for α_2 and α_3, obtained from equations (66) and (77) respectively, are

$$\alpha_2 = 1{\cdot}77848\alpha_1^2, \quad \alpha_3 = 3{\cdot}31102\alpha_1^3, \tag{127}$$

and substitution into equation (126) gives

$$\alpha_4 = 7{\cdot}1573\alpha_1^4. \tag{128}$$

The asymptotic expansion for $f_5(\eta)$ is

$$\begin{aligned} f_5(\eta) \sim & \left\{8\alpha_1(\alpha_2^2+\alpha_1\alpha_3)-\frac{567\pi\alpha_1^5}{80(\frac{1}{4}!)^2}+\tfrac{25}{2}[(\tfrac{1}{4}!)^2F(\tfrac{3}{2})-\tfrac{3}{8}\pi^{\frac{3}{2}}2^{\frac{1}{2}}]\,\alpha_1^5\right\}h_5 \\ & +\left\{\tfrac{1}{45}\alpha_1(-P_1+40\alpha_1^4)-\frac{7\pi^{\frac{1}{2}}\alpha_1^3\alpha_2}{15\,.\,2^{\frac{1}{4}}(\frac{1}{4}!)}\right\}\eta^6-\frac{\pi\alpha_1^3\alpha_2\eta^5}{5(\frac{1}{4}!)^2}-\frac{2\pi\alpha_1^5}{3(\frac{1}{4}!)^2}\eta^4\ln\eta \\ & +\left\{-\tfrac{2}{3}\alpha_1^2\alpha_3+\frac{\pi\alpha_1^5}{(\frac{1}{4}!)^2}[\tfrac{13}{10}-\tfrac{1}{6}(2\ln 2+\gamma-\tfrac{1}{2}\pi)]\right\}\eta^4-\tfrac{16}{3}\alpha_1^3\alpha_2\eta^3 \\ & +\left\{\alpha_5+\frac{21\,.\,\pi^{\frac{1}{2}}\alpha_1^3\alpha_2}{2^{\frac{1}{4}}(\frac{1}{4}!)}+50\alpha_1^5\left[\frac{2^{\frac{3}{2}}(\frac{1}{4}!)^2\phi(\frac{3}{2})}{9\pi^{\frac{1}{2}}}-\frac{2^{\frac{1}{2}}\pi^{\frac{5}{2}}}{32(\frac{1}{4}!)^2}\right]\right\}\eta^2+\frac{4\pi\alpha_1^3\alpha_2}{(\frac{1}{4}!)^2}\eta\ln\eta \\ & +\left\{4(\alpha_2\alpha_3+\alpha_1\alpha_4)-\frac{21\,.\,2^{\frac{1}{2}}\pi^{\frac{5}{2}}\alpha_1^3\alpha_2}{20(\frac{1}{4}!)^4}+\frac{\pi}{(\frac{1}{4}!)^2}(2\ln 2+\gamma-\tfrac{1}{2}\pi-3)\,\alpha_1^3\alpha_2\right\}\eta \\ & +\left(4\alpha_1\alpha_2^2+\frac{449\pi\alpha_1^5}{60(\frac{1}{4}!)^2}\right)-\frac{7\pi^{\frac{1}{2}}\alpha_1^3\alpha_2}{2^{\frac{5}{4}}(\frac{1}{4}!)\,\eta^2}-\frac{\pi\alpha_1^3\alpha_2}{2(\frac{1}{4}!)^2\eta^3}+\frac{37\pi\alpha_1^5}{24(\frac{1}{4}!)^2\eta^4}+\dots \end{aligned} \tag{129}$$

By substituting into this asymptotic expansion the numerical values of the constants, the values of the coefficients of the powers in η can be compared with those obtained by Jones (equation (2·3·6)). The coefficients of (129) were obtained to four decimal places and all the terms from η^6 to η^{-4}, except for η^2, agreed with the terms of the numerical expansion due to Jones to the fourth decimal place. Jones replaced the coefficient of η^2 in (129) by α_5 so that comparison is not possible in this case.

From (43),

$$a_7 = -\frac{2^{\frac{1}{2}}}{252}\left\{8\alpha_1(\alpha_2^2+\alpha_1\alpha_3)-\frac{567\pi\alpha_1^5}{80(\frac{1}{4}!)^2}+\tfrac{25}{2}[(\tfrac{1}{4}!)^2F(\tfrac{3}{2})-\tfrac{3}{8}\pi^{\frac{3}{2}}2^{\frac{1}{2}}]\,\alpha_1^5\right\}. \tag{130}$$

A method of finding $F(\frac{3}{2})$ was discussed earlier and this gave

$$F(\tfrac{3}{2}) = 1{\cdot}2056, \tag{131}$$

so if we substitute the numerical values of α_2, α_3 from (127) and of $F(\frac{3}{2})$ from (131), a_7 becomes

$$a_7 = -0{\cdot}00089\alpha_1^5. \tag{132}$$

It is now appropriate to consider the work of Jones. He introduced the function $\bar{f}_5(\eta)$ where

$$\bar{f}_5(\eta) = f_5(\eta)-\alpha_5\eta^2-4\alpha_1\alpha_4\eta-\lambda\alpha_1^5(1+\tfrac{1}{3}\eta^4-\tfrac{1}{252}\eta^8)+\tfrac{1}{45}\alpha_1P_1\eta^6, \tag{133}$$

and λ is a constant to be determined later. The numerical asymptotic expansion for $\bar{f}_5(\eta)$ was used to start an integration of the differential equation at $\eta = 4{\cdot}0$ towards the origin as far as $\eta = 0{\cdot}5$. He obtained the solution near $\eta = 0$ by fitting the polynomial

$$\bar{f}_5(\eta) = a+b\eta+c\eta^4+d\eta^5 \tag{134}$$

obtained by neglecting terms $O(\eta^8)$. Since in this method only the condition for the absence of exponentially large terms had been satisfied, the condition for a double zero at the origin had still to be satisfied. This was done by ensuring that the series expansion of $f_5(\eta)$ for small η, obtained from equations (133) and (134), did not contain a constant term or a term in η. This gave

$$\alpha_4 = 8{\cdot}1_5\alpha_1^4, \quad \lambda = -0{\cdot}84. \tag{135}$$

The numerical value of α_4 that has been given in equation (128) does not compare favourably with this value. λ is given by the coefficient of h_5 in the asymptotic expansion (129) and, substituting the numerical values of the constants, this gives

$$\lambda = 0{\cdot}16, \tag{136}$$

which again does not give good agreement.

In view of this it was decided to obtain $f_5(\eta)$ as a power series for small η from equation (125) and to compare this with the numerical solution for $\bar{f}_5(\eta)$, given by Jones (table 2), in the range $0{\cdot}5 \leqslant \eta \leqslant 1{\cdot}0$. The series expansion for small η is

$$f_5(\eta) - \alpha_5\eta^2 - 4\alpha_1\alpha_4\eta - \lambda\alpha_1^5(1+\tfrac{1}{3}\eta^4 - \tfrac{1}{252}\eta^8) + \tfrac{1}{45}\alpha_1 P_1\eta^6$$
$$= -\lambda\alpha_1^5 - 4\alpha_1\alpha_4\eta + \tfrac{55}{3}\alpha_1^5\eta^2 - \tfrac{1}{3}\lambda\alpha_1^5\eta^4 - \frac{5\,.\,2^{\frac{1}{4}}\pi^{\frac{3}{2}}\alpha_1^5\eta^5}{12(\frac{1}{4}!)} + O(\eta^8). \tag{137}$$

The left-hand side of this equation is not quite $\bar{f}_5(\eta)$ as used by Jones since he assumed that the asymptotic expansion of $\bar{f}_5(\eta)$ did not contain a term in η^2. Then if his $\bar{f}_5(\eta)$ is given by

$$\bar{f}_5(\eta) = f_5(\eta) - \alpha_5\eta^2 - 4\alpha_1\alpha_4\eta - \lambda\alpha_1^5(1+\tfrac{1}{3}\eta^4 - \tfrac{1}{252}\eta^8) + \tfrac{1}{45}\alpha_1 P_1\eta^6 + \mu\eta^2, \tag{138}$$

the asymptotic expansion (129) for $f_5(\eta)$ leads to

$$\mu = -\frac{21\pi^{\frac{1}{2}}\alpha_1^3\alpha_2}{2^{\frac{1}{4}}(\frac{1}{4}!)} - 50\alpha_1^5\left[\frac{2^{\frac{3}{2}}(\frac{1}{4}!)^2\phi(\frac{3}{2})}{9\pi^{\frac{1}{2}}} - \frac{2^{\frac{1}{2}}\pi^{\frac{5}{2}}}{32(\frac{1}{4}!)^2}\right]. \tag{139}$$

Hence the series expansion of $\bar{f}_5(\eta)$ for small η is

$$\bar{f}_5(\eta) = -\lambda\alpha_1^5 - 4\alpha_1\alpha_4\eta + \left\{\tfrac{55}{3}\alpha_1^5 - 50\alpha_1^5\left[\frac{2^{\frac{3}{2}}(\frac{1}{4}!)^2\phi(\frac{3}{2})}{9\pi^{\frac{1}{2}}} - \frac{2^{\frac{1}{2}}\pi^{\frac{5}{2}}}{32(\frac{1}{4}!)^2}\right] - \frac{21\pi^{\frac{1}{2}}\alpha_1^3\alpha_2}{2^{\frac{1}{4}}(\frac{1}{4}!)}\right\}\eta^2$$
$$-\tfrac{1}{3}\lambda\alpha_1^5\eta^4 - \frac{5\,.\,2^{\frac{1}{4}}\pi^{\frac{3}{2}}\alpha_1^5\eta^5}{12(\frac{1}{4}!)} + O(\eta^8). \tag{140}$$

This indicates that the polynomial (134) should have contained a term in η^2. A method of finding the value of $\phi(\frac{3}{2})$ has been discussed earlier and this gave

$$\phi(\tfrac{3}{2}) = 1{\cdot}36. \tag{141}$$

Substituting the numerical values of α_2, α_3, α_4, λ and $\phi(\frac{3}{2})$ from equations (127), (128), (136) and (141) in equation (140), the series expansion of $\bar{f}_5(\eta)$ is

$$\bar{f}_5(\eta)/\alpha_1^5 = -0{\cdot}16 - 28{\cdot}63\eta - 5{\cdot}86\eta^2 - 0{\cdot}05\eta^4 - 3{\cdot}04\eta^5 + O(\eta^8). \tag{142}$$

The values of $\bar{f}_5(\eta)$ and $\bar{f}_5'(\eta)$ have been computed and compared with Jones's values in the following table:

	η ...	0·5	0·6	0·7	0·8	0·9	1·0
$\frac{\bar{f}_5(\eta)}{\alpha_1^5}$	from equation (142)	−16·0	−19·7	−23·6	−27·8	−32·5	−37·7
	from Jones's table 2	−15·8	−19·5	−23·4	−27·7	−32·4	−37·7
$\frac{\bar{f}_5'(\eta)}{\alpha_1^5}$	from equation (142)	−35·5	−37·7	−40·6	−44·4	−49·3	−55·8
	from Jones's table 2	−35·6	−37·9	−40·9	−44·8	−49·8	−56·4

The first term neglected in the series expansion (142) is

$$\{\tfrac{1}{252}\lambda\,\alpha_1^5 - \tfrac{1}{5040}\alpha_1(61\alpha_2^2 + 52\alpha_1\alpha_3)\}\,\eta^8 \simeq -0{\cdot}07\alpha_1^5\eta^8, \tag{143}$$

which shows that the differences in the values of $\bar{f}_5'$ are of a reasonable magnitude. Thus there is excellent agreement between these results and they confirm the values of α_4 and λ.

4. The equation for $f_6(\eta)$ without suction

In obtaining the differential equation for $f_6(\eta)$ the terms arising from $F_5(\eta)$ and $F_6(\eta)$ will be omitted so that the equation is in the form considered by Goldstein (1948) and Jones (1948). In this case, $f_6(\eta)$ satisfies

$$f_6''' - \tfrac{1}{2}\eta^3 f_6'' + 5\eta^2 f_6' - 9\eta f_6 = (8f_1''f_5 + 4f_1f_5'' - 10f_1'f_5') + (7f_2''f_4 + 5f_2f_4'' - 10f_2'f_4') + (6f_3f_3'' - 5f_3'^2), \tag{144}$$

with boundary conditions $$f_6(0) = 0, \quad f_6'(0) = 0. \tag{145}$$

Substitution of the solutions for f_1, f_2, f_3, f_4 and f_5 from equations (52), (56), (67), (78) and (125) gives for $f_6(\eta)$

$$\begin{aligned} f_6''' &- \tfrac{1}{2}\eta^3 f_6'' + 5\eta^2 f_6' - 9\eta f_6 \\ &= 8\alpha_1^2\alpha_2^2\{5\eta^2 k_4'' - 20\eta k_4' + 14k_4\} + 8\alpha_1^4\alpha_2\{(\tfrac{11}{3}\eta^5 - 40\eta)\,k_4'' + (-\tfrac{86}{3}\eta^4 + 32)\,k_4' + \tfrac{140}{3}\eta^3 k_4\} \\ &\quad + 50\alpha_1^6(4\eta^2 p_5'' - 20\eta p_5' + 16p_5) + \tfrac{64}{9}\alpha_1^6(6k_3k_3'' - 5k_3'^2) + [16\alpha_1^3\alpha_3\eta^2 + \tfrac{16}{3}\alpha_1^4\alpha_2(\tfrac{9}{2}\eta^5 - 28\eta) \\ &\quad + 4\alpha_1^6(\eta^8 - \tfrac{247}{9}\eta^4 + \tfrac{43}{3})]\,k_3'' + [-\tfrac{160}{3}\alpha_1^3\alpha_3\eta + 16\alpha_1^4\alpha_2(-\tfrac{21}{2}\eta^4 + \tfrac{10}{3}) \\ &\quad + \tfrac{4}{9}\alpha_1^6(-63\eta^7 + \tfrac{2021}{2}\eta^3)]\,k_3' + [32\alpha_1^3\alpha_3 + 288\alpha_1^4\alpha_2\eta^3 + 4\alpha_1^6(12\eta^6 - \tfrac{541}{3}\eta^2)]\,k_3 \\ &\quad + 8\alpha_1^2(\alpha_2^2 + \alpha_1\alpha_3)\,(16 - \tfrac{16}{3}\eta^4 - \tfrac{20}{63}\eta^8) + \tfrac{16}{3}\alpha_1^4\alpha_2\eta^7 - \tfrac{26}{45}\alpha_1^2 P_1\eta^6 - \tfrac{1}{3}\alpha_2 P_1(8\eta^3 + \tfrac{2}{5}\eta^7) \\ &\quad + \tfrac{160}{9}\alpha_1^6\eta^6 - \tfrac{10}{3}\alpha_1^2\alpha_4\eta^5 - (8\alpha_1^2\alpha_2^2 + 16\alpha_1^3\alpha_3)\,\eta^4 + \tfrac{320}{3}\alpha_1^4\alpha_2\eta^3 + (128\alpha_1^6 - 16\alpha_1\alpha_5 \\ &\quad - 16\alpha_2\alpha_4 - 8\alpha_3^2)\,\eta^2 - (72\alpha_1\alpha_2\alpha_3 + 12\alpha_2^3 + 16\alpha_1^2\alpha_4)\,\eta - 32\alpha_1^3\alpha_3 - 16\alpha_1^2\alpha_2^2. \end{aligned} \tag{146}$$

In order to consider the work of Goldstein and Jones, this equation is written

$$f_6''' - \tfrac{1}{2}\eta^3 f_6'' + 5\eta^2 f_6' - 9\eta f_6 = -16\alpha_1\alpha_5\eta^2 + \overline{G}_6(\eta). \tag{147}$$

The function $f_6(\eta)$ has the three complementary functions η^2, g_6 and h_6 where h_6 is exponentially large and g_6 is the terminating series

$$g_6 = \eta + \tfrac{1}{15}\eta^5 - \tfrac{1}{1260}\eta^9. \tag{148}$$

The appropriate solution of equation (147) is

$$f_6(\eta) = 4\alpha_1\alpha_5(\eta - g_6) + k_6(\eta), \tag{149}$$

where $k_6(\eta)$ is independent of α_5. From the boundary condition (145) k_6 has a double zero at the origin and since $f_6(\eta)$ must not be exponentially large, k_6 must not be exponentially large. For this, $\overline{G}_6(\eta)$ must satisfy the condition

$$\int_0^\infty (2\eta g_6 - \eta^2 g_6')\,\overline{G}_6 \exp(-\tfrac{1}{8}\eta^4)\,d\eta = 0. \tag{150}$$

Goldstein assumed that this condition was satisfied and Jones, by a numerical method, obtained the value $-4\alpha_1^6 \pm 4\alpha_1^6$ for the integral so that it was possible that the condition (150) might hold. The object of considering the solution for f_6 will be to show that this integral condition is not satisfied.

To evaluate the integral (150) it is desirable to remove those terms on the right-hand side of equation (146) which do not give rise to exponentially large terms in $f_6(\eta)$. From equations (86) and (119) $p_5(\eta)$ satisfies

$$p_5''' - \tfrac{1}{2}\eta^3 p_5'' + \tfrac{9}{2}\eta^2 p_5' - 8\eta p_5 = k_3'. \tag{151}$$

A particular integral for the terms involving $p_5(\eta)$ is obtained by looking for a solution of the form

$$f(\eta) = P(\eta)\,p_5''+Q(\eta)\,p_5'+R(\eta)\,p_5, \tag{152}$$

where $P(\eta)$, $Q(\eta)$ and $R(\eta)$ are polynomials and by using the relation (151) for p_5'''. The previous methods are applied for those terms in k_3, k_4 and powers of η that yield simple particular integrals. These give a general solution

$$\begin{aligned} f_6(\eta) &= \alpha_6\eta^2+\beta_1 h_6+\beta_2 g_6+100\alpha_1^6 p_5'+16\alpha_1^2\alpha_2^2 k_4'+\tfrac{2}{9}\alpha_1^4\alpha_2(-7\eta^2 k_4-\tfrac{17}{2}\eta^3 k_4'+83k_4'') \\ &\quad+\tfrac{16}{3}\alpha_1^3\alpha_3 k_3'+\tfrac{32}{3}\alpha_1^4\alpha_2 k_3''+\tfrac{2}{9}\alpha_1^6(138\eta k_3-107\eta^2 k_3'+6\eta^3 k_3'')+\tfrac{32}{63}\alpha_1^2(\alpha_2^2+\alpha_1\alpha_3)\,(-\eta^7+42\eta^3) \\ &\quad-\tfrac{1}{45}\alpha_2 P_1\eta^6-\tfrac{13}{9}\alpha_1^2 P_1(\tfrac{1}{15}\eta^5+\eta)+\tfrac{8}{9}\alpha_1^4\alpha_2\eta^6+\tfrac{400}{9}\alpha_1^6(\tfrac{1}{15}\eta^5+\eta)-\tfrac{2}{3}\alpha_1^2\alpha_4\eta^4 \\ &\quad-\tfrac{1}{3}(8\alpha_1^2\alpha_2^2+16\alpha_1^3\alpha_3)\,\eta^3+(4\alpha_1\alpha_5+4\alpha_2\alpha_4+2\alpha_3^2-32\alpha_1^6)\,\eta+(8\alpha_1\alpha_2\alpha_3+\tfrac{4}{3}\alpha_2^3)+j_6(\eta), \end{aligned} \tag{153}$$

where α_6, β_1 and β_2 are constants and where $j_6(\eta)$ is a particular integral of

$$j_6''''-\tfrac{1}{2}\eta^3 j_6''+5\eta^2 j_6'-9\eta j_6 = 166\alpha_1^4\alpha_2 k_4'+\tfrac{64}{9}\alpha_1^6(6k_3k_3''-5k_3'^2). \tag{154}$$

By substituting the values of $p_5'(0)$, $k_3'(0)$ and $k_4'(0)$ it can be shown that

$$f_6(0) = j_6(0)+\beta_1-16\alpha_1\alpha_2\alpha_3-8\alpha_1^2\alpha_4-\tfrac{8}{3}\alpha_2^3. \tag{155}$$

Now $f_6(\eta)$ must have a double zero at the origin and must not contain exponentially large terms. Introducing

$$l_6(\eta) = j_6(\eta)+\beta_1 h_6+\beta_3 g_6-16\alpha_1\alpha_2\alpha_3-8\alpha_1^2\alpha_4-\tfrac{8}{3}\alpha_2^3, \tag{156}$$

where β_3 is a constant, we choose β_1 and β_3 so that $l_6(\eta)$ has a double zero at the origin. Thus the first boundary condition of (145) is automatically satisfied and the second boundary condition can be satisfied by choosing the required multiple $(\beta_2-\beta_3)$ of g_6. It follows that the condition that the asymptotic expansion of f_6 does not contain exponentially large terms is the condition that the asymptotic expansion of $l_6(\eta)$ does not contain exponentially large terms. From equations (154) and (156), $l_6(\eta)$ is a particular integral of

$$l_6''''-\tfrac{1}{2}\eta^3 l_6''+5\eta^2 l_6'-9\eta l_6 = 166\alpha_1^4\alpha_2 k_4'+\tfrac{64}{9}\alpha_1^6(6k_3k_3''-5k_3'^2)+(24\alpha_2^3+72\alpha_1^2\alpha_4+144\alpha_1\alpha_2\alpha_3)\,\eta. \tag{157}$$

Then the integral condition (150) becomes

$$\int_0^\infty (\eta^2-\tfrac{1}{5}\eta^6+\tfrac{1}{180}\eta^{10})\,L_6(\eta)\exp(-\tfrac{1}{8}\eta^4)\,d\eta = 0, \tag{158}$$

where $L_6(\eta)$ is the right-hand side of equation (157). Since

$$\int_0^\infty (\eta^2-\tfrac{1}{5}\eta^6+\tfrac{1}{180}\eta^{10})\,\eta\exp(-\tfrac{1}{8}\eta^4)\,d\eta = \tfrac{2}{9}, \tag{159}$$

the contribution of the terms in η to the integral is

$$= 16\alpha_1^2\alpha_4+32\alpha_1\alpha_2\alpha_3+\tfrac{16}{3}\alpha_2^3 \tag{160}$$

and, from equations (127) and (128), this is

$$= 332{\cdot}954\alpha_1^6. \tag{161}$$

The remainder of (158), namely

$$\int_0^\infty \{166\alpha_1^4\alpha_2 k_4'+\tfrac{64}{9}\alpha_1^6(6k_3k_3''-5k_3'^2)\}\,(\eta^2-\tfrac{1}{5}\eta^6+\tfrac{1}{180}\eta^{10})\exp(-\tfrac{1}{8}\eta^4)\,d\eta, \tag{162}$$

was evaluated numerically on the Manchester University Mercury computer. The function $k_3(\eta)$ was obtained by integrating the differential equation (69) for k_3 using the Runge–Kutta fixed step method. The asymptotic expansion (61) was used to find the values of k_3, k_3' and k_3'' at $\eta = 4{\cdot}0$, and the integration was carried back to the origin. This gave a check on the accuracy of the numerical method since at the origin

$$k_3(0) = 1, \quad k_3'(0) = -\frac{3\,.\,2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{10(\frac{1}{4}!)^3} \quad \text{and} \quad k_3''(0) = 0.$$

A further check on these results was made by using the asymptotic expansion (61) for k_3 in the range $2{\cdot}5 \leqslant \eta \leqslant 4{\cdot}0$ and the values were confirmed. The requirement of accurate results meant that a small step length of 0·01 was taken for the differential equation and this was checked by halving the interval. The method for k_4 was similar to this using the asymptotic expansion (62) and the differential equation (81). From these results the values of the integrand were stored at the interval 0·01 and the Euler–Maclaurin formula was applied for the integration.

$$\int_{x_0}^{x_n} f(x)\,\mathrm{d}x = \tfrac{1}{2}(\delta x)\,[f_0+2f_1+2f_2+\ldots+2f_{n-1}+f_n] \\ -\tfrac{1}{12}(\delta x)^2\,[(f_n'-f_0')-\tfrac{1}{60}(\delta x)^2\,(f_n'''-f_0''')+\ldots]. \quad (163)$$

The term $\frac{1}{12}(\delta x)^2 f_n'$ may be neglected so that the integral reduced to the trapezium rule

$$\int_0^4 f(x)\,\mathrm{d}x = \tfrac{1}{2}(\delta x)\,[f_0+2f_1+2f_2+\ldots+2f_{n-1}+f_n]. \quad (164)$$

This gave a value for the integral of

$$(-341{\cdot}578 \pm 0{\cdot}005)\,\alpha_1^6. \quad (165)$$

The value of the integral from $\eta = 4{\cdot}0$ to infinity is negligibly small since it is

$$< \alpha_1^6 \int_4^\infty \eta^{17} \exp\,(-\tfrac{1}{8}\eta^4)\,\mathrm{d}\eta < 0{\cdot}0001\alpha_1^6.$$

Hence from (161) and (165), the value of the left-hand side of equation (158) is

$$(-8{\cdot}62 \pm 0{\cdot}01)\,\alpha_1^6, \quad (166)$$

so that the integral condition is not satisfied. Thus it is necessary to complete the series for ψ by including the terms in $\ln\xi$ as given in equation (26).

5. The functions $F_5(\eta)$ and $F_6(\eta)$

The equation for $F_5(\eta)$ is $\quad F_5''' - \tfrac{1}{2}\eta^3 F_5'' + \tfrac{9}{2}\eta^2 F_5' - 8\eta F_5 = 0, \quad (167)$

with boundary conditions $\quad F_5(0) = F_5'(0) = 0. \quad (168)$

The solution of equation (167) satisfying these boundary conditions is

$$F_5(\eta) = \beta_5 \eta^2, \quad (169)$$

where β_5 is a constant. In equation (37) the term (38), namely $f_0' F_5' - f_0'' F_5$, was omitted and from equations (46) and (169) this is clearly zero.

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The equation for $F_6(\eta)$ is

$$F_6''' - \tfrac{1}{2}\eta^3 F_6'' + 5\eta^2 F_6' - 9\eta F_6 = -16\alpha_1\beta_5\eta^2, \tag{170}$$

with boundary conditions $F_6(0) = F_6'(0) = 0.$ (171)

The solution of (170) satisfying these boundary conditions is

$$F_6(\eta) = 4\alpha_1\beta_5(\eta - g_6) + \beta_6\eta^2, \tag{172}$$

where β_6 is a constant. Substituting for g_6 from equation (148) gives

$$F_6(\eta) = 4\alpha_1\beta_5(\tfrac{1}{1260}\eta^9 - \tfrac{1}{15}\eta^5) + \beta_6\eta^2. \tag{173}$$

If the terms arising from $F_5(\eta)$ and $F_6(\eta)$ are included in the differential equation for $f_6(\eta)$, equation (147) becomes

$$f_6''' - \tfrac{1}{2}\eta^3 f_6'' + 5\eta^2 f_6' - 9\eta f_6 = -16\alpha_1\alpha_5\eta^2 + \bar{G}_6 - 2\alpha_1\beta_5(\eta^2 - \tfrac{1}{5}\eta^6 + \tfrac{1}{180}\eta^{10}). \tag{174}$$

Hence the integral condition for a solution of equation (174) with a double zero at the origin not to be exponentially large is

$$\int_0^\infty \{\bar{G}_6(\eta) - 2\alpha_1\beta_5(\eta^2 - \tfrac{1}{5}\eta^6 + \tfrac{1}{180}\eta^{10})\}(\eta^2 - \tfrac{1}{5}\eta^6 + \tfrac{1}{180}\eta^{10})\exp(-\tfrac{1}{8}\eta^4)\,d\eta = 0, \tag{175}$$

i.e.

$$\tfrac{128}{45}.2^{\frac{3}{4}}(\tfrac{1}{4}!)\,\alpha_1\beta_5 = \int_0^\infty \bar{G}_6(\eta^2 - \tfrac{1}{5}\eta^6 + \tfrac{1}{180}\eta^{10})\exp(-\tfrac{1}{8}\eta^4)\,d\eta. \tag{176}$$

Thus equation (176) determines β_5. All the previous work for $F_5(\eta)$ and $F_6(\eta)$ remains unchanged if there is suction present in the form (35). To evaluate the integral on the right-hand side of equation (176) the case of no suction will be considered. From equation (166) the value of the integral is $(-8{\cdot}62 \pm 0{\cdot}01)\,\alpha_1^6$ so that

$$\beta_5 = (-2{\cdot}00 \pm 0{\cdot}01)\,\alpha_1^5. \tag{177}$$

In principle the solution for further terms of the asymptotic expansion could be carried out indefinitely but it is clear that the equations for the functions become increasingly complicated.

6. Numerical expansions for the skin friction and the velocity distribution near to the separation point

6·1. *The skin friction*

The skin friction $\mu[\partial u/\partial y]_{y=0}$ is related to $[\partial u_1/\partial y_1]_{y_1=0}$ by equations (17) and (18). Now

$$\left(\frac{\partial u_1}{\partial y_1}\right)_{y_1=0} = 2^{\frac{1}{2}}\xi\left[\frac{\partial^2 f_0}{\partial\eta^2} + \xi\frac{\partial^2 f_1}{\partial\eta^2} + \xi^2\frac{\partial^2 f_2}{\partial\eta^2} + \xi^3\frac{\partial^2 f_3}{\partial\eta^2} + \ldots\right]_{\eta=0} \tag{178}$$

and by substituting the solutions for $f_r(\eta)$, this can be written

$$(\partial u_1/\partial y_1)_{y_1=0} = 2^{\frac{3}{4}}[\alpha_1 x_1^{\frac{1}{2}} + \alpha_2 x_1^{\frac{3}{4}} + \alpha_3 x_1 + (\alpha_4 + \tfrac{10}{9}\alpha_1\alpha_2 V_0 2^{\frac{1}{4}})\,x_1^{\frac{5}{4}} + \ldots]. \tag{179}$$

The numerical values for α_2, α_3 and α_4 can be obtained from equations (66), (77) and (124). Hence it follows that

$$(\partial u_1/\partial y_1)_{y_1=0} = 2^{\frac{3}{4}}[\alpha_1 x_1^{\frac{1}{2}} + 1{\cdot}77848\alpha_1^2 x_1^{\frac{3}{4}} + (3{\cdot}31102\alpha_1^3 - 0{\cdot}73330\alpha_1^2 V_0)\,x_1$$
$$+ (7{\cdot}1573\alpha_1^4 - 2{\cdot}6083\alpha_1^3 V_0 + 0{\cdot}1256\alpha_1^2 V_0^2)\,x_1^{\frac{5}{4}} + \ldots]. \tag{180}$$

6·2. *The velocity distribution near to the separation point*

From equations (42) and (43), the velocity distribution in the neighbourhood of the separation point is given by

$$u_1 = \sum_{r=0}^{r=5} a_{r+2} y_1^{r+2} \tag{181}$$

provided higher-order terms are negligible. The a_{r+2} are given by

$$\left.\begin{aligned} 2!\,a_2 &= 1, \\ 3!\,a_3 &= -V_0, \\ 4!\,a_4 &= V_0^2 - 4\alpha_1^2, \\ 5!\,a_5 &= -V_0(V_0^2 - 12\alpha_1^2) - 16{\cdot}2234\alpha_1^3, \\ 6!\,a_6 &= -2P_1 - 42{\cdot}8273\alpha_1^4 - 24\alpha_1^2 V_0^2 + 63{\cdot}0912\alpha_1^3 V_0 + V_0^4, \\ 7!\,a_7 &= -4{\cdot}47\alpha_1^5 + 7P_1 V_0 + 474{\cdot}66\alpha_1^4 V_0 - 152{\cdot}27\alpha_1^3 V_0^2 + 40\alpha_1^2 V_0^3 - V_0^5 + 5V_1. \end{aligned}\right\} \tag{182}$$

It is interesting to note that when $\alpha_1 = 0$, then by relation (34), equations (182) reduce to equations (15).

7. The functions $f_3(\eta)$ and $f_4(\eta)$

From equations (67) and (77)

$$f_3(\eta) = \alpha_1^3 f_{3,0}(\eta) + \alpha_1^2 V_0 2^{\frac{1}{2}} f_{3,1}(\eta) + \tfrac{1}{6}\alpha_1 V_0^2 \eta^4 - \frac{2^{\frac{3}{2}} V_0^3 \eta^6}{6!}, \tag{183}$$

where

$$\left.\begin{aligned} \alpha_1^3 f_{3,0}(\eta) &= \frac{\pi^3}{400(\frac{1}{4}!)^6}(35 - 8.2^{\frac{1}{2}})\,\alpha_1^3\eta^2 + 4\alpha_1\alpha_2\eta - \tfrac{8}{3}\alpha_1^3(1 + \tfrac{1}{4}\eta^4) + \tfrac{8}{3}\alpha_1^3 k_3, \\ \alpha_1^2 f_{3,1}(\eta) &= \tfrac{1}{30}\alpha_1^2\eta^6 - \tfrac{1}{3}\alpha_2\eta^3 - \tfrac{14}{27}\alpha_1^2\eta^2. \end{aligned}\right\} \tag{184}$$

The values of the functions $f_{3,0}(\eta)$ and $f_{3,1}(\eta)$ and their first derivatives are given in table 1.

From equations (78), (77) and (124)

$$f_4(\eta) = \tfrac{1}{6}P_1(\eta^3 - \tfrac{1}{105}\eta^7) + \alpha_1^4 f_{4,0}(\eta) + \alpha_1^3 V_0 2^{\frac{1}{2}} f_{4,1}(\eta) + \alpha_1^2 V_0^2 f_{4,2}(\eta) - \tfrac{1}{30}\alpha_1 V_0^3 2^{\frac{1}{2}}\eta^5 + \tfrac{1}{1260}V_0^4\eta^7, \tag{185}$$

where

$$\left.\begin{aligned} \alpha_1^4 f_{4,0}(\eta) &= \left\{\frac{25.2^{\frac{1}{4}}\pi^{\frac{3}{2}}\alpha_1^4}{14(\frac{1}{4}!)} - \frac{\pi^3(35 - 8.2^{\frac{1}{2}})\,\alpha_1^2\alpha_2}{400(\frac{1}{4}!)^6}\right\}\eta^2 - \tfrac{2}{3}\alpha_1^2\alpha_2\eta^4 + 8\alpha_1^2\alpha_2 k_4 \\ &\qquad + 2\left\{\alpha_2^2 + \frac{\pi^3(35 - 8.2^{\frac{1}{2}})\,\alpha_1^4}{200(\frac{1}{4}!)^6}\right\}\eta + \tfrac{16}{3}\alpha_1^4(k_3' - \eta^3 + \tfrac{1}{84}\eta^7), \\ \alpha_1^3 f_{4,1}(\eta) &= \tfrac{2}{5}\alpha_1^3\eta^5 - \frac{\pi^3(35 - 8.2^{\frac{1}{2}})\,\alpha_1^3\eta^3}{1200(\frac{1}{4}!)^6} - \tfrac{58}{27}\alpha_1\alpha_2\eta^2 + \tfrac{16}{27}\alpha_1^3\eta - \tfrac{4}{27}\alpha_1^3[4\eta k_3 + \eta^2 k_3'], \\ \alpha_1^2 f_{4,2}(\eta) &= -\tfrac{2}{105}\alpha_1^2\eta^7 + \tfrac{1}{6}\alpha_2\eta^4 + \tfrac{28}{81}\alpha_1^2\eta^3 + \frac{13.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}\alpha_1^2\eta^2}{756(\frac{1}{4}!)}. \end{aligned}\right\} \tag{186}$$

The values of the functions $f_{4,0}(\eta)$, $f_{4,1}(\eta)$ and $f_{4,2}(\eta)$ and their first derivatives are given in table 2.

TABLE 1

η	$f_{3,0}(\eta)$	$f'_{3,0}(\eta)$	$f_{3,1}(\eta)$	$f'_{3,1}(\eta)$	η	$f_{3,0}(\eta)$	$f'_{3,0}(\eta)$	$f_{3,1}(\eta)$	$f'_{3,1}(\eta)$
0	0	0	0	0					
0·1	0·033	0·662	−0·006	−0·121	2·1	+1·64	−17·94	−4·92	−1·85
0·2	0·132	1·322	−0·025	−0·279	2·2	−0·46	−24·20	−5·04	−0·58
0·3	0·297	1·975	−0·063	−0·471	2·3	−3·24	−31·60	−5·02	+1·08
0·4	0·527	2·611	−0·121	−0·697	2·4	−6·82	−40·27	−4·81	+3·19
0·5	0·818	3·218	−0·203	−0·957	2·5	−11·34	−50·34	−4·37	5·82
0·6	1·169	3·780	−0·313	−1·247	2·6	−16·94	−61·99	−3·63	9·04
0·7	1·572	4·277	−0·453	−1·564	2·7	−23·80	−75·36	−2·53	12·93
0·8	2·021	4·685	−0·627	−1·902	2·8	−32·08	−90·65	−1·02	17·57
0·9	2·506	4·976	−0·834	−2·256	2·9	−42·00	−108·03	+1·01	23·06
1·0	3·012	5·116	−1·078	−2·616	3·0	−53·76	−127·72	+3·63	29·48
1·1	3·523	5·069	−1·357	−2·971	3·1	−67·63	−149·94	6·94	36·95
1·2	4·018	4·793	−1·672	−3·308	3·2	−83·84	−174·90	11·06	45·58
1·3	4·472	4·239	−2·018	−3·611	3·3	−102·71	−202·86	16·10	55·48
1·4	4·854	3·355	−2·392	−3·862	3·4	−124·52	−234·07	22·20	66·79
1·5	5·130	+2·081	−2·788	−4·038	3·5	−149·64	−268·82	29·51	79·63
1·6	5·256	+0·351	−3·197	−4·115	3·6	−178·41	−307·38	38·18	94·15
1·7	5·182	−1·907	−3·606	−4·063	3·7	−211·25	−350·07	48·40	110·50
1·8	4·854	−4·773	−4·004	−3·850	3·8	−248·58	−397·20	60·35	128·85
1·9	4·205	−8·336	−4·370	−3·438	3·9	−290·85	−449·12	74·24	149·35
2·0	+3·16	−12·69	−4·68	−2·79	4·0	−338·57	−506·18	90·30	172·20

TABLE 2

η	$f_{4,0}(\eta)$	$f'_{4,0}(\eta)$	$f_{4,1}(\eta)$	$f'_{4,1}(\eta)$	$f_{4,2}(\eta)$	$f'_{4,2}(\eta)$	η	$f_{4,0}(\eta)$	$f'_{4,0}(\eta)$	$f_{4,1}(\eta)$	$f'_{4,1}(\eta)$	$f_{4,2}(\eta)$	$f'_{4,2}(\eta)$
0	0	0	0	0	0	0							
0·1	0·072	1·431	−0·020	−0·402	0·002	0·037	2·1	−12·69	−80·27	−0·27	28·59	6·09	4·65
0·2	0·287	2·855	−0·083	−0·869	0·008	0·101	2·2	−21·85	−103·53	+3·15	40·19	6·48	3·08
0·3	0·647	4·255	−0·195	−1·398	0·023	0·201	2·3	−33·54	−131·05	7·86	54·51	6·68	+0·75
0·4	1·141	5·600	−0·364	−1·983	0·050	0·342	2·4	−48·22	−163·39	14·16	71·99	6·60	−2·51
0·5	1·764	6·851	−0·593	−2·612	0·093	0·531	2·5	−66·40	−201·17	22·38	93·12	6·14	−6·92
0·6	2·506	7·951	−0·887	−3·269	0·158	0·774	2·6	−88·66	−245·06	32·92	118·46	5·17	−12·69
0·7	3·347	8·835	−1·247	−3·929	0·250	1·075	2·7	−115·64	−295·79	46·23	148·62	3·55	−20·08
0·8	4·257	9·421	−1·672	−4·563	0·375	1·437	2·8	−148·07	−354·16	62·83	184·29	+1·09	−29·39
0·9	5·213	9·614	−2·158	−5·129	0·539	1·860	2·9	−186·76	−421·04	84·30	226·20	−2·40	−40·94
1·0	6·163	9·302	−2·694	−5·574	0·749	2·341	3·0	−232·59	−497·40	108·31	275·18	−7·18	−55·10
1·1	7·052	8·356	−3·267	−5·836	1·009	2·873	3·1	−286·58	−584·24	138·60	332·14	−13·53	−72·27
1·2	7·808	6·629	−3·853	−5·836	1·325	3·446	3·2	−349·83	−682·71	175·03	398·06	−21·75	−92·89
1·3	8·346	3·952	−4·422	−5·479	1·699	4·041	3·3	−423·55	−794·00	218·55	474·02	−32·23	−117·46
1·4	8·561	+0·135	−4·933	−4·653	2·133	4·634	3·4	−509·10	−919·42	270·21	561·20	−45·39	−146·53
1·5	8·328	−5·037	−5·332	−3·226	2·624	5·193	3·5	−607·96	−1060·38	331·20	660·88	−61·71	−180·68
1·6	7·500	−11·806	−5·553	−1·042	3·169	5·676	3·6	−721·75	−1218·40	402·85	774·43	−81·72	−220·58
1·7	5·905	−20·443	−5·509	+2·080	3·755	6·031	3·7	−852·26	−1395·10	486·60	903·37	−106·04	−266·91
1·8	+3·339	−31·252	−5·099	6·347	4·369	6·192	3·8	−1001·45	−1592·25	584·09	1049·30	−135·35	−320·47
1·9	−0·429	−44·569	−4·194	12·000	4·985	6·081	3·9	−1171·15	−1811·72	697·09	1213·98	−170·40	−382·08
2·0	−5·67	−60·77	−2·64	19·31	5·57	5·60	4·0	−1364·61	−2055·52	827·57	1399·29	−212·06	−452·65

II. A NUMERICAL SOLUTION OF THE LAMINAR BOUNDARY-LAYER EQUATIONS

8. Introduction

In the following sections, a method will be presented for the numerical solution of the laminar boundary-layer equations in the general case with suction so as to enable the solution to be carried step-by-step from the leading edge to the separation point. The previous work which indicated the presence of a singularity at separation will first be discussed.

Hartree (1939*a*) considered a linearly retarded mainstream velocity $U(x) = 1-\frac{1}{8}x$ for flow over an impermeable surface, so that $v_s(x) \equiv 0$. The numerical results suggested that there was a singularity at the separation point because of the behaviour of the skin friction in the neighbourhood of that point and because of the breakdown of the process of integration when an attempt was made to carry it past the separation point. Goldstein (1948) then constructed an asymptotic solution for the immediate neighbourhood of the separation point which has since been modified and extended by Stewartson (1958). A numerical comparison of the solutions of Hartree and Goldstein by Jones (1948) indicated there were no serious discrepancies. Leigh (1955) used an automatic computer for a more accurate investigation of the same case as Hartree, namely $U = 1-\frac{1}{8}x$, $v_s(x) = 0$. He confirmed Hartree's conclusions and obtained agreement with Goldstein's asymptotic solution. Further evidence of the existence of this singularity is that methods of expansion in series (Howarth 1938; Ulrich 1943; Bussmann & Ulrich 1944; Görtler 1955) have not been found to converge in the neighbourhood of the separation point. It is desirable to find whether similar behaviour is encountered in the case of suction and to see if this behaviour agrees with the asymptotic solution that has been obtained earlier.

Leigh has shown it is possible to use an automatic computer to obtain the velocity distribution near the separation point. He took the boundary-layer equation in the form of equation (1) and used a method where given the velocity distribution at a cross-section $x = x_1$, the velocity distribution at a cross-section $x = x_2$ further downstream could be obtained. The most serious difficulty in using the equation in the form (1) is that, in general, this method cannot be started from the leading edge or forward stagnation point. Thus it is necessary to calculate the velocity distribution at a certain distance downstream which would probably be done by a series method. As well as extending the numerical work to include suction, it seemed desirable to produce a method of obtaining the velocity distribution at any cross-section starting from that at the leading edge for any external velocity distribution and for any distribution of suction. This method will be seen to lead to a number of other advantages.

9. The partial differential equation to be numerically integrated

A transformation due to Görtler (1955) is applied to the boundary-layer equation. The velocity potential of the outer flow is

$$\phi = \int_0^x U(x')\,\mathrm{d}x', \tag{187}$$

where $x = 0$ is the leading edge of the surface.

Independent variables are taken as

$$\xi = \frac{\phi}{U_0 l}, \quad \eta = \frac{Uy}{(2\nu\phi)^{\frac{1}{2}}}, \tag{188}$$

where U_0 and l are a suitable reference velocity and length respectively.

The stream function ψ is taken of the form

$$\psi = (2\nu\phi)^{\frac{1}{2}} f(\xi, \eta) \tag{189}$$

and the velocity components become

$$u = U\frac{\partial f}{\partial \eta}, \tag{190}$$

$$v = -\left(\frac{\nu}{2\phi}\right)^{\frac{1}{2}} U\left\{f + 2\xi\frac{\partial f}{\partial \xi} + (\beta - 1)\,\eta\frac{\partial f}{\partial \eta}\right\}, \tag{191}$$

where
$$\beta = \beta(\xi) = \frac{2\phi(\mathrm{d}U/\mathrm{d}x)}{U^2}. \tag{192}$$

When the velocity components are substituted in equation (1), the laminar boundary-layer equation becomes

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} + \beta(\xi)\left\{1 - \left(\frac{\partial f}{\partial \eta}\right)^2\right\} = 2\xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta\,\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right). \tag{193}$$

This equation could also have been obtained from an equation derived by Piercy, Whitehead & Tyler (1948). However, Görtler's approach shows that it is unnecessary to find the whole outer potential flow in advance since the only property needed is the velocity distribution $U(x)$ at the outer edge of the boundary layer.

The boundary conditions (5) and (6) become

$$\left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} = 0, \tag{194}$$

$$f(\xi, 0) + 2\xi\frac{\partial}{\partial \xi} f(\xi, 0) = \left(\frac{2\phi U_0}{lU^2}\right)^{\frac{1}{2}} v_s(x) = K_s(\xi), \tag{195}$$

where $K_s(\xi)$ is a non-dimensional velocity of suction in the (ξ, η) co-ordinates, and as

$$\eta \to \infty, \quad \frac{\partial f}{\partial \eta} \to 1, \quad \frac{\partial^r f}{\partial \eta^r} \to 0 \quad (r > 1). \tag{196}$$

At the leading edge, $\xi = 0$, equation (193) becomes

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + f\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \beta(0)\left\{1 - \left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\right\} = 0, \tag{197}$$

with boundary conditions: at $\eta = 0$, $\mathrm{d}f/\mathrm{d}\eta = 0$

and
$$\left.\begin{aligned} f(0) &= \lim_{\xi\to 0}\left[\tfrac{1}{2}\xi^{-\frac{1}{2}}\int_0^{\xi} \xi_1^{-\frac{1}{2}} K_s(\xi_1)\,\mathrm{d}\xi_1\right] \\ &= \lim_{x\to 0}\left[(2\xi)^{-\frac{1}{2}}\int_0^{x} v_s(x_1)\,\mathrm{d}x_1\right]; \end{aligned}\right\} \tag{198}$$

as $\eta \to \infty$, $\mathrm{d}f/\mathrm{d}\eta \to 1$.

Equation (197) with the boundary conditions (198) can be regarded as a special case of the equation of similar profiles, namely

$$\frac{d^3f}{d\eta^3}+\alpha f\frac{d^2f}{d\eta^2}+\beta\left\{1-\left(\frac{df}{d\eta}\right)^2\right\}+\gamma\frac{d^2f}{d\eta^2}=0, \tag{199}$$

where α, β and γ are constants with boundary conditions

$$\text{at}\quad \eta=0,\quad f=\frac{\partial f}{\partial\eta}=0\quad\text{and}\quad\text{as}\quad \eta\to\infty,\quad \frac{df}{d\eta}\to 1. \tag{200}$$

Numerous solutions of the equation of similar profiles have been obtained by Schlichting & Bussman (1943), Thwaites (1949), Brown & Donoughe (1951), Hartree (1937) and the author (§ 16).

The method of solution will be such that given $\partial f/\partial\eta$ at the cross-section $\xi=\xi_1$, then $\partial f/\partial\eta$ at the cross-section $\xi=\xi_2$ can be obtained. Thus it can be started at the leading edge, and, in fact the initial distribution of $\partial f/\partial\eta$ is given by a solution of the equation of similar profiles.

This form of the laminar boundary-layer equation has several other advantages. First, the boundary-layer thickness changes slowly in terms of the η variable as compared with the y variable. Secondly, as the iterative procedure uses the function $\partial f/\partial\eta$ which satisfies the condition $\partial f/\partial\eta\to 1$ as $\eta\to\infty$ for all ξ, it implies that only the inner part of the boundary layer requires correcting.

Another difficulty in using the form (1) was found by Hartree (1939*a*). He observed that there were two alternative replacements for dp/dx, namely, either

$$\frac{1}{2}\left[\left(\frac{dp}{dx}\right)_1+\left(\frac{dp}{dx}\right)_2\right], \tag{201}$$

or

$$\frac{p_2-p_1}{x_2-x_1}=-\frac{\frac{1}{2}(U_2^2-U_1^2)}{(x_2-x_1)}, \tag{202}$$

where the suffixes 1 and 2 denote the step from x_1 to x_2. At the surface, (201) appears to be the correct replacement but as $\eta\to\infty$ (202) becomes the right expression to use. In the case of the linearly decreasing mainstream velocity $U=1-\frac{1}{8}x$ used by Hartree (1939*a*) and Leigh (1955), these two approximations become identical but this will obviously not be true in general. Hartree (1939*b*), in treating Schubauer's observed pressure distribution for an elliptic cylinder, used a suitable combination of these two expressions. The term in equation (193) corresponding to dp/dx is $\beta(\xi)$ and this is multiplied by a term $\{1-(\partial f/\partial\eta)^2\}$ which tends to zero as $\eta\to\infty$. Thus the difference between replacements corresponding to (201) and (202) is negligible as $\eta\to\infty$ so that the required approximation is

$$\frac{1}{2}\left[\beta(\xi_1)\left\{1-\left(\frac{\partial f}{\partial\eta}\right)^2\right\}_{\xi=\xi_1}+\beta(\xi_2)\left\{1-\left(\frac{\partial f}{\partial\eta}\right)^2\right\}_{\xi=\xi_2}\right]. \tag{203}$$

10. Method of solution

Equation (193) can be rearranged as

$$\frac{\partial^2 q}{\partial\eta^2}+\left[\int_0^\eta\left(q+2\xi\frac{\partial q}{\partial\xi}\right)d\eta+K_s(\xi)\right]\frac{\partial q}{\partial\eta}-2\xi q\frac{\partial q}{\partial\xi}=-\beta(\xi)\,(1-q^2), \tag{204}$$

where

$$q=\partial f/\partial\eta. \tag{205}$$

An appropriate method for the solution of certain types of parabolic partial differential equations was proposed by Hartree & Womersley (1937) and has been found to lead to a stable numerical process. In this derivatives in the ξ direction are replaced by differences and all other quantities by averages.

Thus equation (204) becomes

$$\frac{1}{2}\left\{\frac{d^2q_1}{d\eta^2}+\frac{d^2q_2}{d\eta^2}\right\}+\left[\int_0^\eta\left\{\tfrac{1}{2}(q_1+q_2)+\frac{(\xi_1+\xi_2)}{(\xi_2-\xi_1)}(q_2-q_1)\right\}d\eta\right.$$
$$\left.+\tfrac{1}{2}\{K_s(\xi_1)+K_s(\xi_2)\}\right]\frac{1}{2}\left(\frac{dq_1}{d\eta}+\frac{dq_2}{d\eta}\right)-\frac{(\xi_1+\xi_2)}{(\xi_2-\xi_1)}(q_2-q_1)\,\tfrac{1}{2}(q_2+q_1)$$
$$=-\tfrac{1}{2}[\beta(\xi_1)\,(1-q_1^2)+\beta(\xi_2)\,(1-q_2^2)], \qquad (206)$$

where the suffixes 1 and 2 denote the values of the functions at the cross-sections $\xi=\xi_1$ and $\xi=\xi_2$ respectively. It is supposed that q_1 is known so that q_2 is determined from this equation. Following a similar method to Leigh (1955), take

$$v=q_1+q_2, \qquad (207)$$

then equation (206) becomes

$$\frac{d^2v}{d\eta^2}+\left[\int_0^\eta\{\tfrac{1}{2}v+\lambda(v-2q_1)\}\,d\eta+\tfrac{1}{2}\{K_s(\xi_1)+K_s(\xi_2)\}\right]\frac{dv}{d\eta}-\lambda(v-2q_1)\,v$$
$$=-\beta(\xi_1)\,(1-q_1^2)-\beta(\xi_2)\,\{1-(v-q_1)^2\}, \qquad (208)$$

where
$$\lambda=(\xi_1+\xi_2)/(\xi_2-\xi_1). \qquad (209)$$

The boundary conditions (194) and (196) become

$$\text{at}\quad \eta=0,\quad v=0;\quad \text{as}\quad \eta\to\infty,\quad v\to 2. \qquad (210)$$

(208) is a third-order non-linear differential equation for v which will be solved by an iterative process. If $v^{(m)}$ defines the mth iterative approximation to the solution of this equation then $v^{(m+1)}$ is given by

$$\frac{d^2v^{(m+1)}}{d\eta^2}+\left[\int_0^\eta\{\tfrac{1}{2}v^{(m+1)}+\lambda(v^{(m+1)}-2q_1)\}\,d\eta+\tfrac{1}{2}\{K_s(\xi_1)+K_s(\xi_2)\}\right]\frac{dv^{(m)}}{d\eta}-\lambda(v^{(m+1)}-2q_1)\,v^{(m)}$$
$$=-\beta(\xi_1)\,(1-q_1^2)-\beta(\xi_2)\,\{1-(v^{(m)}-q_1)^2\}. \qquad (211)$$

A rectangular mesh of dimensions $\{(x_2-x_1),h\}$ is now formed by introducing differences in the η-direction. If the suffix j refers to the jth mesh-point in this direction, equation (211), after rearrangement, becomes

$$\left(\frac{d^2v^{(m+1)}}{d\eta^2}\right)_j-\lambda v_j^{(m)}v_j^{(m+1)}+\left[(\tfrac{1}{2}+\lambda)\int_0^{jh}v^{(m+1)}\,d\eta\right]\left(\frac{dv^{(m)}}{d\eta}\right)_j$$
$$=-\beta(\xi_1)\,(1-q_{1,j}^2)-\beta(\xi_2)\,\{1-(v_j^{(m)}-q_{1,j})^2\}-2\lambda q_{1,j}\,v_j^{(m)}$$
$$+\left[2\lambda\int_0^{jh}q_1\,d\eta-\tfrac{1}{2}\{K_s(\xi_1)+K_s(\xi_2)\}\right]\left(\frac{dv^{(m)}}{d\eta}\right)_j. \qquad (212)$$

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The following approximations are now made

$$\left.\begin{aligned}\left\{\frac{\mathrm{d}^2v^{(m+1)}}{\mathrm{d}\eta^2}\right\}_j &= \frac{1}{h^2}\{v_{j+1}^{(m+1)}-2v_j^{(m+1)}+v_{j-1}^{(m+1)}\}+O(1),\\ \left\{\frac{\mathrm{d}v^{(m)}}{\mathrm{d}\eta}\right\}_j &= \frac{1}{2h}\{v_{j+1}^{(m)}-v_{j-1}^{(m)}\}+O(h),\\ \int_0^{jh} v^{(m+1)}\,\mathrm{d}\eta &= h\{v_1^{(m+1)}+v_2^{(m+1)}+\ldots+v_{j-1}^{(m+1)}+\tfrac{1}{2}v_j^{(m+1)}\}+O(h^3),\\ \int_0^{jh} q_1\,\mathrm{d}\eta &= h\{q_{1,1}+q_{1,2}+\ldots+q_{1,j-1}+\tfrac{1}{2}q_{1,j}\}+O(h^3) = h\delta_{1,j}+O(h^3),\end{aligned}\right\} \tag{213}$$

and equation (212) becomes

$$\begin{aligned}v_{j+1}^{(m+1)}-2v_j^{(m+1)}+v_{j-1}^{(m+1)}-\lambda h^2v_j^{(m)}v_j^{(m+1)}+(\tfrac{1}{2}+\lambda)\,\tfrac{1}{4}h^2(v_{j+1}^{(m)}-v_{j-1}^{(m)})\\ \times(v_1^{(m+1)}+v_2^{(m+1)}+\ldots+v_{j-1}^{(m+1)}+\tfrac{1}{2}v_j^{(m+1)})\\ =-h^2\beta(\xi_1)\,(1-q_{1,j}^2)-h^2\beta(\xi_2)\,\{1-(v_j^{(m)}-q_{1,j})^2\}-\tfrac{1}{4}h\{K_s(\xi_1)+K_s(\xi_2)\}\\ \times(v_{j+1}^{(m)}-v_{j-1}^{(m)})-2\lambda h^2q_{1,j}v_j^{(m)}+\lambda h^2\delta_{1,j}(v_{j+1}^{(m)}-v_{j-1}^{(m)}),\end{aligned} \tag{214}$$

so that $\quad v_{j+1}^{(m+1)}+A_j^{(m)}v_j^{(m+1)}+(1+\alpha_j^{(m)})\,v_{j-1}^{(m+1)}+\alpha_j^{(m)}(v_{j-2}^{(m+1)}+\ldots+v_1^{(m+1)}) = C_j^{(m)}, \qquad (215)$

where

$$\left.\begin{aligned}\alpha_j^{(m)} &= \tfrac{1}{4}(1+2\lambda)\,h^2(v_{j+1}^{(m)}-v_{j-1}^{(m)}),\\ A_j^{(m)} &= -2-\lambda h^2v_j^{(m)}+\tfrac{1}{2}\alpha_j^{(m)},\\ \text{and}\quad C_j^{(m)} &= -h^2\beta(\xi_1)\,(1-q_{1,j}^2)-h^2\beta(\xi_2)\,\{1-(v_j^{(m)}-q_{1,j})^2\}\\ &\quad -\tfrac{1}{4}h\{K_s(\xi_1)+K_s(\xi_2)\}\,(v_{j+1}^{(m)}-v_{j-1}^{(m)})-2\lambda h^2q_{1,j}v_j^{(m)}+\lambda h^2\delta_{1,j}(v_{j+1}^{(m)}-v_{j-1}^{(m)}).\end{aligned}\right\} \tag{216}$$

Since $v = 0$ at $\eta = 0$, the first equation is

$$v_2^{(m+1)}+A_1^{(m)}v_1^{(m+1)} = C_1^{(m)}. \tag{217}$$

It is assumed that the Nth mesh-point in the η-direction is sufficiently far from the surface for the outer boundary condition to be satisfied to the required accuracy beyond that point. Then $v_{N+1}^{(m+1)} = 2$ and the Nth equation becomes

$$A_N^{(m)}v_N^{(m+1)}+(1+\alpha_N^{(m)})\,v_{N-1}^{(m+1)}+\alpha_N^{(m)}(v_{N-2}^{(m+1)}+\ldots+v_1^{(m+1)}) = C_N^{(m)}-2. \tag{218}$$

The set of simultaneous linear equations (215), (217) and (218) can be written in matrix form

$$\mathbf{A}^{(m)}\mathbf{v}^{(m+1)} = \mathbf{C}^{(m)}, \tag{219}$$

where $\quad\mathbf{v}^{(m+1)}$ is the column vector $\{v_1^{(m+1)}, v_2^{(m+1)}, \ldots, v_N^{(m+1)}\}$,

$\mathbf{C}^{(m)}$ is the column vector $\{C_1^{(m)}, C_2^{(m)}, \ldots, C_{N-1}^{(m)}, C_N^{(m)}-2\}$,

and $\quad\mathbf{A}^{(m)}$ is the matrix

$$\begin{pmatrix} A_1^{(m)} & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots\\ 1+\alpha_2^{(m)} & A_2^{(m)} & 1 & 0 & 0 & \cdots & \cdots & \cdots\\ \alpha_3^{(m)} & 1+\alpha_3^{(m)} & A_3^{(m)} & 1 & 0 & \cdots & \cdots & \cdots\\ \alpha_4^{(m)} & \alpha_4^{(m)} & 1+\alpha_4^{(m)} & A_4^{(m)} & 1 & \cdots & \cdots & \cdots\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & A_{N-2}^{(m)} & 1 & 0\\ \alpha_{N-1}^{(m)} & \alpha_{N-1}^{(m)} & \cdots & \cdots & \alpha_{N-1}^{(m)} & 1+\alpha_{N-1}^{(m)} & A_{N-1}^{(m)} & 1\\ \alpha_N^{(m)} & \alpha_N^{(m)} & \cdots & \cdots & \alpha_N^{(m)} & \alpha_N^{(m)} & 1+\alpha_N^{(m)} & A_N^{(m)}\end{pmatrix}. \tag{220}$$

The problem of finding $\mathbf{v}^{(m+1)}$ will be solved by treating the more general problem of finding the inverse of the matrix $\mathbf{A}^{(m)}$.

11. Inversion of the matrix

If there are N intervals $\mathbf{A}^{(m)}$ contains N^2 elements, but to determine the matrix completely only the $2N$ values of $A_j^{(m)}$ and $\alpha_j^{(m)}$ are required. Thus the storing of the elements of $\mathbf{A}^{(m)}$ presents no problems. Since the matrix $\mathbf{A}^{(m)}$ is very nearly lower triangular it would appear that the best method of obtaining a solution is to put (219) into the form $\mathbf{L}\mathbf{v}^{(m+1)} = \mathbf{C}^{(m)*}$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{C}^{(m)*}$ is a column vector. Then the inversion of $\mathbf{L}$ is a simple procedure.

A method similar to this was first tried. $v_1^{(m+1)}$ is treated as an unknown and, by successive substitution in equations (217) and (215), $v_2^{(m+1)}$, $v_3^{(m+1)}$, ..., $v_N^{(m+1)}$, in that order, are found in terms of $v_1^{(m+1)}$. Then $v_1^{(m+1)}$ is determined by equation (218) and hence all the $v_j^{(m+1)}$. However it appeared that although the first value $v_1^{(m+1)}$ was excellent and the first few values of $v_j^{(m+1)}$ were very good, the difference between $v_j^{(m+1)}$ and its expected value increased with j at an alarming rate. Reducing the size of the interval in the ξ direction only had the effect of increasing this deviation further. This is similar to the behaviour encountered by Hartree (1939*a*) when, using a step-by-step method of integration, he found that his solution was very sensitive to small changes in the initial conditions at the surface.

How this behaviour arises in this method is clear. Suppose that λ is large, which if $\xi_1 \neq 0$, is equivalent to $\xi_2 - \xi_1$ small. Now $v_{j+1}^{(m+1)}$ is determined from $(v_1^{(m+1)} \dots v_j^{(m+1)})$ and, from equation (215), this is the difference between two terms involving λ. If λ is large, these two terms are relatively large so that their difference will lead to errors. Further, if there is an error ϵ_1 in $v_j^{(m+1)}$, it can be seen from equation (215) that the error ϵ_2 introduced into $v_{j+1}^{(m+1)}$ is $[2+\lambda h^2 v_j^{(m)} - \frac{1}{8}(1+2\lambda)\, h^2(v_{j+1}^{(m)} - v_{j-1}^{(m)})]\,\epsilon_1$ so that $\epsilon_2 > (1+\lambda h^2)\,\epsilon_1$ for large j and in the outer part of the boundary layer, where $v_j^{(m)} \simeq 2 \gg (v_{j+1}^{(m)} - v_{j-1}^{(m)})$, this error $\simeq 2(1+\lambda h^2)\,\epsilon_1$. Thus the error produced grows rapidly as j increases. It is analogous with what occurs in some of the solutions of the equation of similar profiles (equation (199)). In these small changes in $f''(0)$ cause large changes in $f'(\eta)$ as $n \to \infty$ and a method of overcoming this is to integrate back from infinity. A similar method is applicable here. Then if $v_j^{(m+1)}$ is found in terms of $(v_{j+1}^{(m+1)} \dots v_N^{(m+1)})$ it is to be expected that the values of $v_j^{(m+1)}$ will be accurate and, further, that they will become more accurate as λ increases. This is done by transforming the matrix equation (219) to the form

$$\mathbf{U}^{(m)}\mathbf{v}^{(m+1)} = \mathbf{C}^{(m)*}, \tag{221}$$

where $\mathbf{U}^{(m)}$ is an upper triangular matrix which can be obtained by Choleski's method (Hartree 1952).

The matrix $\mathbf{A}^{(m)}$ is resolved into the product of a lower triangular matrix $\mathbf{L}^{(m)}$ and an upper triangular matrix $\mathbf{U}^{(m)}$ where

$$\mathbf{L}^{(m)} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ l_{21} & 1 & 0 & \dots \\ l_{31} & l_{32} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \text{and} \quad \mathbf{U}^{(m)} = \begin{pmatrix} u_{11} & 1 & 0 & 0 & \dots \\ 0 & u_{22} & 1 & 0 & \dots \\ 0 & 0 & u_{33} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{222}$$

Then, from equations (219) and (221),

$$\mathbf{C}^{(m)*} = \{\mathbf{L}^{(m)}\}^{-1}\,\mathbf{C}^{(m)} \tag{223}$$

and the elements of $\mathbf{C}^{(m)*}$ are given by

$$C_1^{(m)*} = C_1^{(m)}, \quad C_r^{(m)*} = C_r^{(m} - \sum_{j=1}^{r-1} l_{rj} C_j^{(m)*} \quad (2 \leqslant j \leqslant N). \tag{224}$$

The $v_j^{(m+1)}$ are given by equation (221) so that

$$v_N^{(m+1)} = \frac{C_N^{(m)*}}{u_{N,N}}, \quad v_j^{(m+1)} = \frac{C_j^{(m)*} - v_{j+1}^{(m+1)}}{u_{j,j}} \quad (1 \leqslant j \leqslant N-1). \tag{225}$$

The solution confirmed that the values of $v_j^{(m+1)}$ obtained by this method were good and their accuracy increased as λ increased. It is interesting that a matrix so very nearly lower triangular had to be transformed to an upper triangular matrix so that a well-behaved solution could be obtained.

12. Programming

The program has ideally been written for an accuracy of four to five decimals in q. Obviously a solution for fewer decimal places may be obtained, but in this case a smaller program would be adequate. The integration is carried out using an initial estimate of $2q_{1,j}$ for $v_j^{(0)}$ in three stages, namely

(i) for an interval $h = 0{\cdot}2$ where η takes the values

$$\eta = 0{\cdot}2\,(0{\cdot}2)\,6{\cdot}4;$$

(ii) for an interval $h = 0{\cdot}1$ where η takes the values

$$\eta = 0{\cdot}1\,(0{\cdot}1)\,6{\cdot}4;$$

(iii) for an interval $h = 0{\cdot}05$ where η takes the values

$$\eta = 0{\cdot}05\,(0{\cdot}05)\,3{\cdot}2.$$

A truncation error arises from using a finite interval in the η-direction and its leading term may be removed by Richardson's h^2-extrapolation. If v_h is the value obtained by integrating at an interval h and v_{2h} is the value at an interval $2h$, then Richardson's h^2-extrapolation gives a better solution as $v_R = v_h + \frac{1}{3}(v_h - v_{2h})$. The outer limit of $\eta = 6{\cdot}4$ was taken because it was expected that at that point the outer boundary condition would be attained to the required number of decimal places (four or five) for all cross-sections from the leading edge to separation. The results from (i) and (ii), after h^2-extrapolation, gave a solution for q_2 in the range $0{\cdot}1\,(0{\cdot}1)\,6{\cdot}4$ and the values for the range $3{\cdot}2\,(0{\cdot}1)\,6{\cdot}4$ were to the required accuracy (six decimal places). These also gave the outer boundary condition $v(3{\cdot}25)$ at $\eta = 3{\cdot}25$ for routine (iii) in which the right-hand side of equation (218) and the last element of $\mathbf{C}^{(m)}$ are replaced by $C_N^{(m)} - v(3{\cdot}25)$. The results from (ii) and (iii), after h^2-extrapolation, gave the solution for q_2 in the range $0{\cdot}05\,(0{\cdot}05)\,3{\cdot}2$ and these were checked with those obtained from (i) and (ii) in this range. By this method it was found that the error due to using a finite interval in the η-direction was negligible. Also since there is very little difference between routines (ii) and (iii), it is simple to adapt one routine for them both.

Richardson's h^2-extrapolation only removes the leading term in the truncation error and in the neighbourhood of the separation point, in the ξ-direction, the coefficients of higher-order terms are large so that it does not remove the largest term in the error. Thus it was not used in the ξ-direction and the error was kept small by halving the interval and ensuring that the results for the full interval and for the two half-intervals agreed to the required number of decimals. This was found to be completely satisfactory.

The function λ, given by equation (209), is important in determining the convergence of the solution. As pointed out earlier, decreasing the step in the ξ-direction increases λ and the accuracy of the results. It is also found that when λ increases the iterations converge more speedily. The average number of iterations required for an accuracy of six decimals was about five for each routine but in the neighbourhood of separation these increased to six, eight and ten for (i), (ii) and (iii) respectively. Near separation the number of iterations could be reduced by using better estimates for $v_j^{(0)}$—either by extrapolation from the previous steps or by using the solution obtained from (i) for routines (ii) and (iii). However it was found that the number of iterations was only slightly reduced near separation and that $v_j^{(0)} = 2q_{1,j}$ was the best estimate near the leading edge so that for simplicity this estimate was used throughout. The time for a complete checked step from ξ_1 to ξ_2 was about twenty minutes. The whole program has been written so that if the routines associated with the particular problem have been inserted and given the distribution of q at the leading edge and the first step, the program will automatically carry on to the separation point. However, as this takes a long time, a facility for outputting the distribution of q at the end of every step has been inserted so that the program may be conveniently restarted.

The difficulty of starting at the leading edge has not completely disappeared since when $\xi_1 = 0$ equation (209) gives $\lambda = 1$. Thus in taking a step from the leading edge the error cannot be reduced by halving the interval. This difficulty is overcome by starting away from the leading edge at a cross-section $\xi = \xi_0$ where the distribution of q is the same as that at the leading edge to the required number of decimal places. If there is a known series solution, the position of ξ_0 may be obtained by considering the magnitude of its terms. Alternatively a large step may be taken from $\xi = 0$ and the magnitude of the change in q estimated so that a position can be calculated for a negligible change in q. Both these methods worked well. The first step from ξ_0 will, in general, be small but it was found that successive steps increased by a factor of 2 or 3 so that the solution is soon away from the leading edge. It remains to consider the particular routines required for any specific problem.

13. Routines required for a particular boundary-layer flow

If the mainstream velocity $U(x)$ is given by

$$U(x) = U_0 f(x'), \tag{226}$$

then the relations between the co-ordinates (ξ, η) of Görtler's equation and the non-dimensional co-ordinates (x', y') usually associated with the boundary-layer equation are

$$\xi = \int_0^{x'} f(x')\,\mathrm{d}x' = g(x'), \quad \eta = \left\{\frac{f(x')}{[2g(x')]^{\frac{1}{2}}}\right\} y'. \tag{227}$$

The only other functions in equation (215) that require determination are $\beta(\xi)$ and $K_s(\xi)$. From equations (192) and (227), $\beta(\xi)$ is given by

$$\beta(\xi) = \frac{2g(x')\,\mathrm{d}\{f(x')\}/\mathrm{d}x'}{[f(x')]^2}, \tag{228}$$

and, from equations (195) and (227),

$$K_s(\xi) = \frac{[2g(x')]^{\frac{1}{2}}}{f(x')}\, v_s(x'). \tag{229}$$

Clearly the functions ξ, $\beta(\xi)$ and $K_s(\xi)$ that are required in equation (215) can be obtained from equations (227), (228) and (229). Since, in the numerical solution, when the step in ξ is halved it is desirable to find x', the routine for the inverse function $x' = g^{-1}(\xi)$ is also wanted. Thus for any boundary-layer flow only the routines for ξ, $\beta(\xi)$, $K_s(\xi)$ and $g^{-1}(\xi)$ are needed. In the case of symmetric flow past a circular cylinder which will be considered later, if the external velocity distribution is given by $U(x) = U_0 \sin x'$, the subroutines required are

$$\left.\begin{aligned} \xi &= 1-\cos x', & \beta(\xi) &= \sec^2 \tfrac{1}{2}x' \cos x', \\ K_s(\xi) &= v_s(x') \sec \tfrac{1}{2}x', & g^{-1}(x') &= \sin^{-1}(x'). \end{aligned}\right\} \tag{230}$$

A routine has been included in the program to compute $\partial^2 f/\partial\eta^2$ at $\eta = 0$, the displacement thickness δ_1 and the momentum thickness δ_2. The value of $(\partial^2 f/\partial\eta^2)_{\eta=0}$ is obtained by using finite differences for q near $\eta = 0$. The displacement thickness δ_1 and momentum thickness δ_2 are

$$\delta_1 = \int_0^\infty \left(1-\frac{u}{U}\right) dy = \frac{(2\phi\nu)^{\frac{1}{2}}}{U} \int_0^\infty \left(1-\frac{\partial f}{\partial\eta}\right) d\eta, \tag{231}$$

$$\delta_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right) dy = \frac{(2\phi\nu)^{\frac{1}{2}}}{U} \int_0^\infty \frac{\partial f}{\partial\eta}\left(1-\frac{\partial f}{\partial\eta}\right) d\eta. \tag{232}$$

The integrals in equations (231) and (232) were evaluated by using the Euler–Maclaurin formula given in equation (163). For the interval $h = 0{\cdot}05$ and for an accuracy of five decimals these reduced to

$$\int_0^\infty \left(1-\frac{\partial f}{\partial\eta}\right) d\eta = \tfrac{1}{2}h[1+2(1-q_1)+\ldots+2(1-q_{n-1})+(1-q_n)] - \tfrac{1}{12}h^2\left(\frac{\partial^2 f}{\partial\eta^2}\right)_{\eta=0}, \tag{233}$$

$$\int_0^\infty \frac{\partial f}{\partial\eta}\left(1-\frac{\partial f}{\partial\eta}\right) d\eta = \tfrac{1}{2}h[2q_1(1-q_1)+\ldots+2q_{n-1}(1-q_{n-1})+q_n(1-q_n)] + \tfrac{1}{12}h^2\left(\frac{\partial^2 f}{\partial\eta^2}\right)_{\eta=0}. \tag{234}$$

As $(\partial^2 f/\partial\eta^2)_{\eta=0}$ has already been evaluated, it is simple to obtain the values of these integrals.

In the cases considered, the accuracy of the results was checked by the momentum equation which is

$$\frac{\tau_\omega}{\rho U^2} = \frac{d\delta_2}{dx} + \frac{(\delta_1+2\delta_2)}{U}\frac{dU}{dx} + \frac{1}{U}\left\{\left(\frac{U_0\nu}{l}\right)^{\frac{1}{2}} v_s(x)\right\}, \tag{235}$$

where τ_ω is the skin friction given by

$$\tau_\omega = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \rho U^2\left(\frac{\nu}{2\phi}\right)^{\frac{1}{2}}\left(\frac{\partial^2 f}{\partial\eta^2}\right)_{\eta=0}. \tag{236}$$

Then $d\delta_2/dx'$ can be obtained from equation (235) and compared with the results for δ_2 as a function of x'. It should be noted that when $U(x)$ is nearly zero the value of $d\delta_2/dx'$ obtained from equation (235) will not be accurate. This explains the errors in the values of $d\delta_2/dx'$ near the leading edge given in tables 4 and 8.

To summarize, a program has been written to carry the solution of the laminar boundary-layer equation from the leading edge to the separation point in which for any particular flow only: (i) the subroutines for ξ, $\beta(\xi)$, $K_S(\xi)$ and $g^{-1}(\xi)$, and (ii) the solution of the equation of similar profiles for the distribution of q_1 at the leading edge and the initial step from ξ_1 to ξ_2 are required.

14. The potential flow past a circular cylinder with constant suction

14·1. *The numerical results*

The potential flow past a circular cylinder with mainstream velocity $U = U_0 \sin x'$ and non-dimensional velocity of suction $v_s(x) = 0{\cdot}5$ was chosen because the results could be compared with those obtained by Bussmann & Ulrich (1944).

The solution was taken so that q_{2j} was accurate to four decimal places after each step. This accuracy was determined by ensuring that the solution at the full interval and at two consecutive half-intervals in the ξ-direction had a maximum difference of less than $\pm 5\times 10^{-5}$. However, as the value from the integration at the half-intervals was taken, it may be presumed that the maximum error in q_j due to a finite step is 10^{-5}. Thus although it is to be expected that the error would build up after a large number of steps had been taken, nevertheless after estimating the errors from all these steps, it appears that q_j should be accurate to four decimal places at separation. This accuracy has been obtained without assuming anything about the solution becoming more accurate as the separation point is approached due to the smoothing out of irregularities. It was noted earlier that the solution must not be started from the leading edge and the first step taken was from $x'_1 = 0{\cdot}01$ to $x'_2 = 0{\cdot}02$.

Some of the results for $(\partial f/\partial \eta)$ at typical cross-sections have been given in table 3. These have been given in the co-ordinates (x', η) as they are convenient for tabular form. The velocity u at (x', η) is given by $U(\partial f/\partial \eta) = U_0 \sin x'(\partial f/\partial \eta)$ and the factor $\sin x'$ has been tabulated beneath the table. The co-ordinate η can be converted into the non-dimensional boundary layer co-ordinate y' by multiplying by the factor $y'/\eta = \sec \frac{1}{2}x'$, which has also been tabulated below the table.

The values of $(\partial u/\partial y')_{y'=0}$, δ_1, δ_2, $H = \delta_1/\delta_2$ and $\mathrm{d}\delta_2/\mathrm{d}x'$ obtained from the momentum equation are given in table 4. All the values have been included so that the number and size of steps taken can be seen.

The position of separation obtained was 114·7°, whereas Bussman & Ulrich gave 120·9°. To check the solution and to see why there is such a large difference in the position of the separation point, the values of $(\partial u/\partial y')_{y'=0}$ and δ_1 obtained by the two methods have been compared.

Bussmann & Ulrich gave a series expansion for $(\partial u/\partial y)_{y=0}$:

$$(\partial u/\partial y)_{y=0} = 1{\cdot}5418x - 0{\cdot}5303x^3 + 0{\cdot}0545x^5 - 0{\cdot}00316x^7 + 0{\cdot}00004406x^9, \qquad (237)$$

and for δ_1:

$$\delta_1 = (1/\sin x)\{0{\cdot}5419x - 0{\cdot}0012x^3 + 0{\cdot}00419x^5 - 0{\cdot}000691x^7 + 0{\cdot}0001351x^9\} \qquad (238)$$

(where the prime denoting non-dimensional x and y has been and, henceforth, will be omitted). The coefficient of x^7 in equation (238) appears to be in error; this term should read $+0{\cdot}000697x^7$. By substituting for x, the values of $(\partial u/\partial y)_{y=0}$ and δ_1 have been found and compared with those given in table 4:

x (in radians)	0	0·4	1·0	1·5	1·89
$(\partial u/\partial y)_{y=0}$ (computed)	0	0·5833	1·0629	0·8833	0·3507
$(\partial u/\partial y)_{y=0}$ (B. & U.)	0	0·5833	1·0629	0·8845	0·3895
δ_1 (computed)	0·5423	0·5570	0·6499	0·8684	1·4167
δ_1 (B. & U., corrected)	0·5419	0·5566	0·6485	0·8599	1·2822
δ_1 (B. & U., uncorrected)	0·5419	0·5566	0·6468	0·8360	1·1573

TABLE 3

η \ x	0	0·6	1·3	1·5	1·69	1·79	1·89	1·94	1·988	2·0005	2·001637
0·05	0·0749	0·0734	0·0651	0·0593	0·0503	0·0429	0·0320	0·0239	0·0114	0·0042	0·0018
0·10	0·1455	0·1427	0·1271	0·1163	0·0994	0·0856	0·0649	0·0494	0·0255	0·0117	0·0069
0·15	0·2119	0·2080	0·1861	0·1711	0·1473	0·1278	0·0985	0·0766	0·0423	0·0223	0·0154
0·20	0·2742	0·2694	0·2423	0·2236	0·1940	0·1696	0·1329	0·1052	0·0615	0·0359	0·0268
0·25	0·3327	0·3271	0·2957	0·2739	0·2394	0·2109	0·1678	0·1351	0·0830	0·0521	0·0412
0·30	0·3874	0·3812	0·3463	0·3220	0·2836	0·2517	0·2031	0·1660	0·1066	0·0709	0·0581
0·35	0·4384	0·4318	0·3942	0·3681	0·3264	0·2917	0·2387	0·1979	0·1320	0·0919	0·0776
0·40	0·4860	0·4791	0·4395	0·4120	0·3679	0·3311	0·2744	0·2305	0·1590	0·1151	0·0992
0·45	0·5304	0·5232	0·4823	0·4538	0·4080	0·3696	0·3101	0·2637	0·1874	0·1401	0·1229
0·50	0·5715	0·5642	0·5227	0·4936	0·4467	0·4071	0·3456	0·2972	0·2169	0·1667	0·1484
0·60	0·6451	0·6378	0·5962	0·5670	0·5195	0·4791	0·4154	0·3646	0·2788	0·2240	0·2039
0·7	0·7079	0·7009	0·6609	0·6326	0·5862	0·5464	0·4828	0·4314	0·3429	0·2852	0·2638
0·8	0·7611	0·7546	0·7173	0·6906	0·6467	0·6085	0·5469	0·4963	0·4076	0·3487	0·3266
0·9	0·8060	0·8000	0·7660	0·7416	0·7009	0·6652	0·6069	0·5583	0·4716	0·4129	0·3907
1·0	0·8434	0·8382	0·8078	0·7858	0·7490	0·7163	0·6622	0·6166	0·5335	0·4763	0·4545
1·2	0·9002	0·8962	0·8733	0·8565	0·8278	0·8019	0·7579	0·7197	0·6478	0·5964	0·5764
1·4	0·9381	0·9353	0·9190	0·9069	0·8860	0·8667	0·8331	0·8031	0·7448	0·7017	0·6846
1·6	0·9627	0·9608	0·9499	0·9416	0·9271	0·9135	0·8892	0·8671	0·8225	0·7885	0·7747
1·8	0·9782	0·9770	0·9700	0·9646	0·9550	0·9459	0·9293	0·9137	0·8814	0·8560	0·8455
2·0	0·9876	0·9869	0·9826	0·9793	0·9732	0·9674	0·9565	0·9461	0·9238	0·9058	0·8983
2·2	0·9932	0·9928	0·9902	0·9882	0·9846	0·9810	0·9742	0·9676	0·9530	0·9408	0·9356
2·4	0·9964	0·9961	0·9947	0·9936	0·9915	0·9894	0·9853	0·9812	0·9720	0·9642	0·9608
2·6	0·9981	0·9980	0·9972	0·9966	0·9954	0·9942	0·9919	0·9895	0·9840	0·9791	0·9770
2·8	0·9991	0·9990	0·9986	0·9983	0·9976	0·9970	0·9957	0·9944	0·9912	0·9883	0·9870
3·0	0·9996	0·9995	0·9993	0·9991	0·9988	0·9985	0·9978	0·9971	0·9953	0·9937	0·9929
3·2	0·9998	0·9998	0·9997	0·9996	0·9994	0·9993	0·9989	0·9985	0·9976	0·9967	0·9963
3·4	0·9999	0·9999	0·9999	0·9998	0·9997	0·9997	0·9995	0·9993	0·9988	0·9983	0·9981
3·6	1·0000	1·0000	0·9999	0·9999	0·9999	0·9998	0·9998	0·9997	0·9994	0·9992	0·9991
3·8	1·0000	1·0000	1·0000	1·0000	0·9999	0·9999	0·9999	0·9999	0·9997	0·9996	0·9996
4·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	0·9999	0·9999	0·9998	0·9998
sin x	0	0·5646	0·9636	0·9975	0·9929	0·9761	0·9495	0·9326	0·9143	0·9091	0·9086
sec $\frac{1}{2}x$	1	1·047	1·256	1·367	1·506	1·599	1·707	1·769	1·834	1·852	1·853

TABLE 4

x (radians)	$(\partial u/\partial y)_{y=0}$	δ_1	δ_2	H	dδ_2/dx	x (radians)	$(\partial u/\partial y)_{y=0}$	δ_1	δ_2	H	dδ_2/d
0·015000	0·0231	0·5423	0·2497	2·172	0·0096	1·790000	0·5264	1·1917	0·4938	2·413	0·525
0·020000	0·0308	0·5423	0·2497	2·172	0·0090	1·844575	0·4354	1·3002	0·5248	2·477	0·610
0·030000	0·0462	0·5425	0·2497	2·172	0·0026	1·890000	0·3507	1·4167	0·5545	2·555	0·697
0·060000	0·0924	0·5426	0·2498	2·172	0·0069	1·917070	0·2946	1·5035	0·5742	2·618	0·758
0·091211	0·1402	0·5431	0·2501	2·172	0·0071	1·940000	0·2422	1·5930	0·5923	2·690	0·817
0·120000	0·1841	0·5436	0·2503	2·172	0·0105	1·954860	0·2046	1·6625	0·6047	2·749	0·859
0·200000	0·3041	0·5459	0·2513	2·173	0·0165	1·960000	0·1906	1·6897	0·6091	2·774	0·874
0·315750	0·4703	0·5514	0·2537	2·174	0·0264	1·970000	0·1617	1·7482	0·6181	2·829	0·906
0·400000	0·5833	0·5570	0·2562	2·174	0·0332	1·979978	0·1292	1·8178	0·6273	2·898	0·940
0·600000	0·8147	0·5767	0·2647	2·178	0·0530	1·988000	0·0987	1·8869	0·6348	2·972	0·970
0·695921	0·9029	0·5895	0·2704	2·180	0·0638	1·991497	0·0833	1·9234	0·6384	3·013	0·984
0·790000	0·9727	0·6048	0·2769	2·184	0·0758	1·994000	0·0709	1·9535	0·6407	3·049	0·994
0·899408	1·0315	0·6260	0·2862	2·188	0·0920	1·996499	0·0567	1·9885	0·6433	3·091	1·004
1·000000	1·0629	0·6499	0·2962	2·194	0·1094	1·998000	0·0468	2·0135	0·6447	3·123	1·010
1·079123	1·0719	0·6718	0·3055	2·199	0·1251	1·999499	0·0350	2·0440	0·6462	3·163	1·017
1·150000	1·0680	0·6945	0·3149	2·205	0·1414	2·000000	0·0302	2·0563	0·6469	3·179	1·019
1·226013	1·0511	0·7223	0·3264	2·213	0·1616	2·000900	0·0196	2·0840	0·6477	3·218	1·023
1·300000	1·0222	0·7538	0·3392	2·223	0·1845	2·001400	0·0108	2·1074	0·6483	3·251	1·026
1·350716	0·9953	0·7782	0·3490	2·230	0·2025	2·001437	0·0097	2·1103	0·6483	3·255	1·026
1·400000	0·9637	0·8047	0·3594	2·239	0·2222	2·001470	0·0088	2·1125	0·6483	3·258	1·026
1·450152	0·9261	0·8348	0·3712	2·249	0·2448	2·001530	0·0070	2·1176	0·6484	3·266	1·027
1·500000	0·8833	0·8684	0·3840	2·261	0·2704	2·001565	0·0056	2·1210	0·6484	3·271	1·027
1·550026	0·8350	0·9065	0·3982	2·276	0·2998	2·001600	0·0039	2·1257	0·6484	3·278	1·027
1·600000	0·7814	0·9499	0·4141	2·294	0·3338	2·001608	0·0034	2·1270	0·6484	3·280	1·027
1·649901	0·7225	0·9997	0·4318	2·315	0·3731	2·001615	0·0029	2·1283	0·6484	3·282	1·027
1·690000	0·6711	1·0455	0·4475	2·336	0·4096	2·001630	0·0015	2·1321	0·6484	3·288	1·027
1·744734	0·5969	1·1151	0·4698	2·373	0·4654	2·001635	0·0007	2·1341	0·6484	3·291	1·027

It can be seen that agreement is very good in the region from the leading edge to about 70° ($x \simeq 1{\cdot}2$) but then the results become widely different. At $x = 1{\cdot}89$, $(\partial u/\partial y)_{y=0}$ given by (237) is

$$(\partial u/\partial y)_{y=0} = 2{\cdot}9140 - 3{\cdot}5802 + 1{\cdot}3143 - 0{\cdot}2722 + 0{\cdot}0136 = 0{\cdot}3895, \qquad (239)$$

and clearly from the magnitude of these terms, the difference can be accounted for by noting that there are too few terms in the series (237) and (238). Bussmann & Ulrich estimated the position of the separation point by putting $(\partial u/\partial y)_{y=0} = 0$ in (237) but if there is a singularity at separation, it is doubtful whether the series converges near separation. However, even if the series does converge, there are not enough terms in (237) to determine the separation point accurately. Finally the check on the momentum equation by Bussmann & Ulrich was very good up to about 80° ($x \simeq 1{\cdot}4$), but from there to separation it became progressively worse. If $d\delta_2/dx$ is calculated from the values of δ_2 in table 4 and compared with the values obtained from the momentum equation, it is found that there is excellent agreement for all positions to separation. Thus the check implies that the results are very accurate and that the method has worked well. The flow in the upstream neighbourhood of separation will now be considered in greater detail.

14·2. *The solution near separation*

From equations (16), (17) and (18), for the external velocity distribution $U = U_0 \sin x$, dp_1/dx_1 is given by

$$\frac{dp_1}{dx_1} = -U_1\frac{dU_1}{dx_1} = -\frac{U\,dU/dx}{U_S\,dU_S/dx} = -\frac{\sin 2x}{\sin 2x_S}, \qquad (240)$$

and this can be written

$$dp_1/dx_1 = -\cos(2x_1 \tan x_S) - \cot 2x_S \sin(2x_1 \tan x_S). \qquad (241)$$

Then from equation (20), the P_r are given by

$$P_{2r} = \frac{(-1)^r (2\tan x_S)^{2r}}{(2r)!}, \quad P_{2r+1} = \frac{(-1)^r (2\tan x_S)^{2r+1} \cot 2x_S}{(2r+1)!}. \qquad (242)$$

In particular $$P_0 = 1, \quad P_1 = 1 - \tan^2 x_S. \qquad (243)$$

From table 4, the position of separation is

$$x_S = 2{\cdot}00164 \text{ radians}. \qquad (244)$$

The asymptotic expansion for the velocity gradient in the neighbourhood of the separation point has been given in equation (180).

Now V_0, x_1 and $\partial u_1/\partial y_1$ are given by

$$V_0 = v_0(-\sec x_S)^{\frac{1}{2}} = 0{\cdot}77377, \qquad (245)$$

$$x_1 = \frac{x_S - x}{(-\tan x_S)} = \frac{2{\cdot}00164 - x}{2{\cdot}17562}, \qquad (246)$$

and $$\frac{\partial u_1}{\partial y_1} = \frac{1}{\sin x_S(-\cos x_S)^{\frac{1}{2}}}\left(\frac{\partial u}{\partial y}\right) \qquad (247)$$

respectively. Substituting into equation (179)

$$\frac{1}{(8\sin x_S)^{\frac{1}{2}}\,(-\cos x_S)\,(x_S-x)^{\frac{1}{2}}}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \alpha_1+\alpha_2 x_1^{\frac{1}{4}}+\alpha_3 x_1^{\frac{1}{2}}+(\alpha_4+\tfrac{10}{9}\alpha_1\alpha_2 V_0 2^{\frac{1}{2}})\,x_1^{\frac{3}{4}}+\dots \quad (248)$$

and after substituting the numerical values of the coefficients this reduces to

$$\frac{0\cdot 88811}{(2\cdot 00164-x)^{\frac{1}{2}}}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \alpha_1+1\cdot 4644\alpha_1^2(2\cdot 00164-x)^{\frac{1}{4}} + (2\cdot 2448\alpha_1^3-0\cdot 3847\alpha_1^2)\,(2\cdot 00164-x)^{\frac{1}{2}} + (3\cdot 9951\alpha_1^4-1\cdot 1241\alpha_1^3+0\cdot 0420\alpha_1^2)\,(2\cdot 00164-x)^{\frac{3}{4}}+\dots. \quad (249)$$

The numerical solution can be compared with the asymptotic solution by substituting for x and $(\partial u/\partial y)_{y=0}$ at points in the upstream neighbourhood of separation. From this comparison, a value for α_1 can be obtained and equation (249) was programmed on the machine so as to determine α_1. In table 5, the second column gives the value of the left-hand side of equation (249) and the succeeding columns give values of the right-hand side when α_1 is 0·5, 0·55 and 0·6.

Hence equating the left- and right-hand sides of equation (249)

$$\alpha_1 = 0\cdot 555\pm 0\cdot 005. \quad (250)$$

In figure 1, $(\partial u/\partial y)_{y=0}$ has been plotted against (x_S-x) for the numerical results and for equation (249) when $\alpha_1 = 0\cdot 5$, 0·55 and 0·6. This clearly shows the agreement between the numerical results and the asymptotic solution near to separation.

Another comparison between the numerical solution and the asymptotic solution is possible by considering the velocity profile near separation. The velocity distribution close to separation has been given in equations (181) and (182) so that

$$u_1 = \left(\frac{\partial f}{\partial \eta}\right)_{x=x_S} = \sum_{r=0}^{4} a_{r+2}\,y_1^{r+2}, \quad (251)$$

where the a_{r+2} are given in equation (182). The term $a_7 y_1^7$ has been omitted since a_7 is complicated and the terms in (251) are sufficient to determine α_1 to two decimal places. When x_1 is small, the neglected terms in (251) arising from $F_5(\eta)$ and $F_6(\eta)$ are dominated by

$$\tfrac{1}{1120}\alpha_1\beta_5 y_1^8 \ln x_1. \quad (252)$$

Since the value of β_5 with suction has not been obtained, the value of β_5 without suction given in equation (177) is taken. If y_1 is taken to be in the range $0\leqslant y_1\leqslant 0\cdot 8$, then for $\alpha_1 = 0\cdot 555$ and $x_1 = 10^{-6}$ (252) has a maximum value less than 10^{-4} and so is negligible.

From equations (17) and (188)

$$y_1 = \{\sec^2\tfrac{1}{2}x_S-2\}^{\frac{1}{2}}\,\eta = 1\cdot 19756\eta. \quad (253)$$

In table 6 the velocity profile obtained numerically and the velocity profiles for different values of α_1 computed from equations (251) and (253) are given. It can be seen that there is excellent agreement between the velocity profile obtained numerically and those obtained from the asymptotic expansion. Comparison between these profiles gives

$$\alpha_1 = 0\cdot 555\pm 0\cdot 005, \quad (254)$$

which confirms the value (250) obtained by a different method.

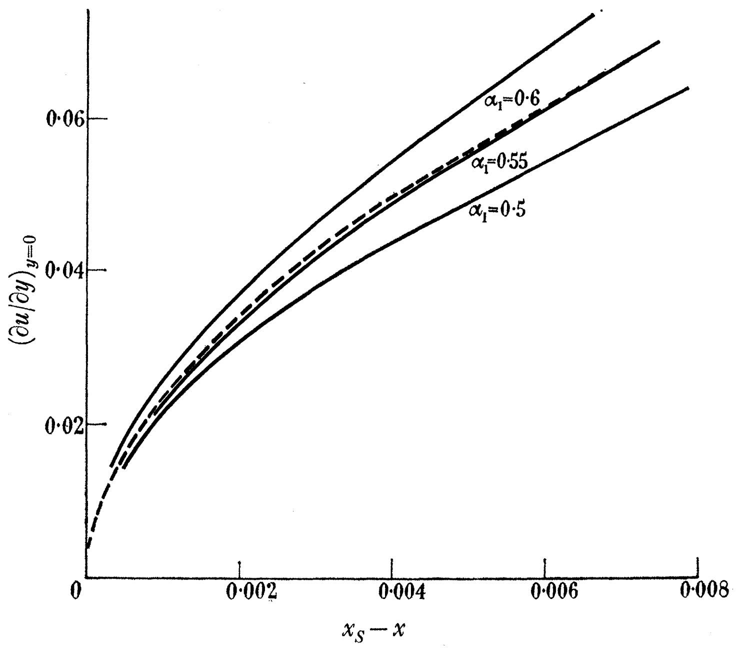

FIGURE 1. Variation of the skin friction with the distance from the separation point. ---, numerical results; ——, asymptotic solutions.

TABLE 5

x	left-hand side of (249)	right-hand side of (249) $\alpha_1 = 0{\cdot}5$	$\alpha_1 = 0{\cdot}55$	$\alpha_1 = 0{\cdot}6$
1·994	0·720	0·635	0·719	0·806
1·9965	0·703	0·619	0·698	0·781
1·998	0·690	0·607	0·683	0·762
1·9995	0·672	0·591	0·663	0·737
2·0	0·663	0·584	0·654	0·726
2·0005	0·653	0·576	0·644	0·713
2·0009	0·642	0·568	0·633	0·700
2·0014	0·615	0·549	0·610	0·672
2·001437	0·609	0·547	0·607	0·669
2·00147	0·606	0·545	0·605	0·666

TABLE 6. THE VELOCITY PROFILE NEAR SEPARATION

η	numerical results	$\alpha_1 = 0{\cdot}5$	$\alpha_1 = 0{\cdot}6$	$\alpha_1 = 0{\cdot}55$	$\alpha_1 = 0{\cdot}555$	$\alpha_1 = 0{\cdot}56$
0·05	0·0018	0·0018	0·0018	0·0018	0·0018	0·0018
0·10	0·0069	0·0069	0·0069	0·0069	0·0069	0·0069
0·15	0·0154	0·0154	0·0154	0·0154	0·0154	0·0154
0·20	0·0268	0·0269	0·0268	0·0268	0·0268	0·0268
0·25	0·0412	0·0412	0·0411	0·0412	0·0412	0·0411
0·30	0·0581	0·0583	0·0580	0·0582	0·0581	0·0581
0·35	0·0776	0·0779	0·0773	0·0776	0·0776	0·0775
0·40	0·0992	0·0998	0·0987	0·0993	0·0992	0·0992
0·45	0·1229	0·1238	0·1221	0·1230	0·1229	0·1229
0·50	0·1484	0·1498	0·1472	0·1486	0·1485	0·1483
0·55	0·1755	0·1776	0·1737	0·1758	0·1756	0·1754
0·60	0·2039	0·2071	0·2015	0·2045	0·2042	0·2039
0·65	0·2334	0·2380	0·2302	0·2344	0·2340	0·2336

15. The potential flow past a circular cylinder with zero suction

15·1. *The numerical results*

The potential flow past a circular cylinder with mainstream velocity $U = U_0 \sin x$ and no suction was next considered. Some of the results for $\partial f/\partial\eta$ at typical cross-sections have been given in table 7 and for $(\partial u/\partial y)_{y=0}$, δ_1, δ_2, H and $\mathrm{d}\delta_2/\mathrm{d}x$ obtained from the momentum equation in table 8.

The position of separation obtained was 104·45°, whereas Ulrich (1943), by a series method, obtained 110·0°. To check the solution and to see why there is such a large difference in the position of the separation point, the values of $(\partial u/\partial y)_{y=0}$ obtained by the two methods have been compared.

Ulrich (1943) gave a series expansion for $(\partial u/\partial y)_{y=0}$ similar to (237) as far as x^9. However, Tifford (1954) has given some results from which more accurate values of the coefficients of the powers of x can be obtained and from which the term in x^{11} can also be calculated. Using these, it is found that, neglecting terms $O(x^{13})$,

$$(\partial u/\partial y)_{y=0} = 1{\cdot}2325877x - 0{\cdot}4829649x^3 + 0{\cdot}0515987x^5 - 0{\cdot}0032351x^7 - 0{\cdot}0000354x^9 - 0{\cdot}0000204x^{11}. \qquad (255)$$

The values of $(\partial u/\partial y)_{y=0}$ obtained from (255) have been compared with those given in table 8 in the following table:

x (radians)	0·5	1·0	1·5	1·6	1·8
$(\partial u/\partial y)_{y=0}$ (computed)	0·5575	0·7979	0·5520	0·4396	0·1049
$(\partial u/\partial y)_{y=0}$ (from (255))	0·5575	0·7979	0·5523	0·4421	0·1588

Agreement is very good in the region from the leading edge to about 80° ($x \simeq 1{\cdot}4$) but then the results deviate as x increases. At $x = 1{\cdot}8$, $(\partial u/\partial y)_{y=0}$ given by (255) is

$$\begin{aligned}(\partial u/\partial y)_{y=0} &= 2{\cdot}21866 - 2{\cdot}81665 + 0{\cdot}97499 - 0{\cdot}19806 - 0{\cdot}00702 - 0{\cdot}01311 \\ &= 0{\cdot}1588,\end{aligned} \qquad (256)$$

and clearly, if the series converges, there are insufficient terms to give $(\partial u/\partial y)_{y=0}$ accurately near the separation point. Ulrich found that the momentum equation was well satisfied up to about 70° ($x \simeq 1{\cdot}2$) but from there to separation agreement was not good whereas if $\mathrm{d}\delta_2/\mathrm{d}x$ is calculated from the values of δ_2 in table 8 and compared with $\mathrm{d}\delta_2/\mathrm{d}x$ obtained from the momentum equation, it is found that there is excellent agreement for all positions from the leading edge to separation. Thus the results appear to be very accurate and 104·45° the correct position of separation.

15·2. *The solution near separation*

From table 8 the position of separation is

$$x_S = 1{\cdot}822983. \qquad (257)$$

The asymptotic expansion for the velocity gradient in the neighbourhood of separation can be obtained from equations (248) and (180). Hence

$$\frac{1}{(8\sin x_S)^{\frac{1}{4}}(-\cos x_S)\,(x_S - x)^{\frac{1}{2}}}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \alpha_1 + 1{\cdot}77848\alpha_1^2 x_1^{\frac{1}{4}} + 3{\cdot}31102\alpha_1^3 x_1^{\frac{1}{2}} + 7{\cdot}1573\alpha_1^4 x_1^{\frac{3}{4}} + \ldots \qquad (258)$$

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Table 7

η \ x	0	0·5	1·0	1·3	1·5	1·65	1·73	1·78	1·816	1·821	1·822983
0·05	0·0604	0·0588	0·0532	0·0459	0·0377	0·0279	0·0201	0·0132	0·0052	0·0030	0·0008
0·10	0·1183	0·1153	0·1045	0·0907	0·0750	0·0561	0·0411	0·0276	0·0119	0·0076	0·0034
0·15	0·1737	0·1695	0·1542	0·1345	0·1119	0·0848	0·0630	0·0434	0·0202	0·0138	0·0075
0·20	0·2266	0·2214	0·2021	0·1771	0·1485	0·1139	0·0857	0·0604	0·0301	0·0217	0·0133
0·25	0·2771	0·2710	0·2482	0·2187	0·1847	0·1432	0·1094	0·0786	0·0415	0·0311	0·0207
0·30	0·3252	0·3184	0·2926	0·2592	0·2204	0·1729	0·1338	0·0980	0·0544	0·0421	0·0297
0·35	0·3710	0·3635	0·3353	0·2985	0·2556	0·2027	0·1589	0·1184	0·0688	0·0546	0·0403
0·40	0·4145	0·4064	0·3763	0·3367	0·2903	0·2327	0·1846	0·1399	0·0845	0·0685	0·0523
0·45	0·4557	0·4472	0·4155	0·3737	0·3244	0·2628	0·2109	0·1624	0·1015	0·0838	0·0659
0·50	0·4946	0·4859	0·4530	0·4095	0·3579	0·2929	0·2377	0·1856	0·1198	0·1005	0·0808
0·6	0·5663	0·5573	0·5231	0·4774	0·4227	0·3527	0·2923	0·2345	0·1598	0·1376	0·1147
0·7	0·6299	0·6209	0·5866	0·5404	0·4844	0·4116	0·3478	0·2857	0·2039	0·1791	0·1534
0·8	0·6859	0·6772	0·6438	0·5984	0·5426	0·4690	0·4034	0·3384	0·2512	0·2243	0·1964
0·9	0·7351	0·7268	0·6950	0·6514	0·5971	0·5244	0·4583	0·3919	0·3010	0·2726	0·2428
1·0	0·7779	0·7702	0·7405	0·6994	0·6476	0·5771	0·5120	0·4455	0·3525	0·3230	0·2919
1·2	0·8467	0·8404	0·8158	0·7811	0·7363	0·6733	0·6132	0·5496	0·4574	0·4272	0·3949
1·4	0·8968	0·8919	0·8727	0·8451	0·8085	0·7554	0·7030	0·6459	0·5597	0·5306	0·4991
1·6	0·9323	0·9288	0·9145	0·8935	0·8651	0·8227	0·7793	0·7306	0·6542	0·6278	0·5987
1·8	0·9568	0·9543	0·9441	0·9289	0·9079	0·8755	0·8414	0·8018	0·7374	0·7145	0·6889
2·0	0·9732	0·9715	0·9646	0·9540	0·9391	0·9154	0·8897	0·8590	0·8072	0·7883	0·7669
2·2	0·9839	0·9827	0·9782	0·9712	0·9610	0·9444	0·9259	0·9031	0·8633	0·8483	0·8312
2·4	0·9905	0·9899	0·9870	0·9825	0·9758	0·9647	0·9518	0·9356	0·9063	0·8950	0·8819
2·6	0·9946	0·9942	0·9925	0·9897	0·9855	0·9783	0·9698	0·9587	0·9380	0·9298	0·9202
2·8	0·9970	0·9968	0·9958	0·9941	0·9916	0·9871	0·9816	0·9744	0·9603	0·9546	0·9479
3·0	0·9984	0·9983	0·9977	0·9968	0·9953	0·9926	0·9892	0·9846	0·9755	0·9717	0·9671
3·2	0·9992	0·9991	0·9988	0·9983	0·9974	0·9959	0·9939	0·9911	0·9854	0·9829	0·9800
3·4	0·9996	0·9996	0·9994	0·9991	0·9987	0·9978	0·9967	0·9950	0·9915	0·9900	0·9881
3·6	0·9998	0·9998	0·9997	0·9996	0·9993	0·9989	0·9982	0·9973	0·9953	0·9944	0·9932
3·8	0·9999	0·9999	0·9999	0·9998	0·9997	0·9994	0·9991	0·9986	0·9975	0·9970	0·9963
4·0	1·0000	1·0000	0·9999	0·9999	0·9998	0·9997	0·9996	0·9993	0·9987	0·9984	0·9980
4·4	1·0000	1·0000	1·0000	1·0000	1·0000	0·9999	0·9999	0·9998	0·9997	0·9996	0·9995
4·8	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	0·9999	0·9999	0·9999
sin x	0	0·4794	0·8415	0·9635	0·9975	0·9969	0·9874	0·9782	0·9701	0·9689	0·9683
sec $\frac{1}{2}x$	1	1·032	1·139	1·256	1·366	1·474	1·542	1·589	1·625	1·630	1·632

Table 8

x (radians)	$(\partial u/\partial y)_{y=0}$	δ_1	δ_2	H	$d\delta_2/dx$	x (radians)	$(\partial u/\partial y)_{y=0}$	δ_1	δ_2	H	$d\delta_2/dx$
0·010000	0·0123	0·6479	0·2923	2·217	0·0131	1·680000	0·3309	1·5029	0·5732	2·622	0·6253
0·020000	0·0246	0·6479	0·2924	2·216	0·0012	1·730000	0·2509	1·6599	0·6068	2·736	0·7188
0·040000	0·0493	0·6481	0·2925	2·216	0·0023	1·738700	0·2356	1·6931	0·6130	2·762	0·7372
0·080000	0·0984	0·6487	0·2927	2·216	0·0072	1·747000	0·2205	1·7270	0·6193	2·789	0·7555
0·160000	0·1952	0·6512	0·2937	2·217	0·0165	1·760000	0·1958	1·7853	0·6294	2·837	0·7859
0·300000	0·3569	0·6595	0·2971	2·220	0·0332	1·780000	0·1540	1·8916	0·6455	2·930	0·8367
0·500000	0·5575	0·6813	0·3061	2·226	0·0578	1·787481	0·1368	1·9390	0·6519	2·975	0·8573
0·664631	0·6839	0·7093	0·3176	2·234	0·0821	1·794000	0·1208	1·9851	0·6576	3·019	0·8761
0·800000	0·7550	0·7408	0·3303	2·243	0·1062	1·800000	0·1049	2·0326	0·6628	3·067	0·8941
0·903958	0·7870	0·7714	0·3425	2·252	0·1282	1·804997	0·0905	2·0778	0·6673	3·114	0·9097
1·000000	0·7979	0·8057	0·3560	2·263	0·1524	1·809000	0·0779	2·1188	0·6710	3·158	0·9228
1·100000	0·7897	0·8494	0·3727	2·279	0·1826	1·812998	0·0639	2·1661	0·6748	3·210	0·9363
1·200000	0·7611	0·9031	0·3927	2·300	0·2197	1·816000	0·0519	2·2083	0·6775	3·259	0·9469
1·251119	0·7386	0·9356	0·4045	2·313	0·2420	1·819000	0·0377	2·2606	0·6805	3·322	0·9579
1·300000	0·7120	0·9702	0·4169	2·327	0·2660	1·821000	0·0256	2·3070	0·6823	3·381	0·9655
1·345233	0·6831	1·0063	0·4296	2·343	0·2910	1·822000	0·0175	2·3392	0·6834	3·423	0·9696
1·390000	0·6503	1·0463	0·4432	2·361	0·3187	1·822450	0·0125	2·3589	0·6838	3·450	0·9714
1·430113	0·6174	1·0864	0·4565	2·380	0·3466	1·822800	0·0072	2·3808	0·6841	3·480	0·9729
1·450081	0·5998	1·1081	0·4636	2·390	0·3617	1·822825	0·0066	2·3832	0·6843	3·483	0·9731
1·470000	0·5814	1·1311	0·4709	2·402	0·3777	1·822850	0·0060	2·3857	0·6843	3·486	0·9732
1·500000	0·5520	1·1685	0·4826	2·421	0·4035	1·822900	0·0047	2·3910	0·6843	3·494	0·9734
1·560000	0·4874	1·2551	0·5086	2·468	0·4629	1·822950	0·0030	2·3984	0·6843	3·505	0·9737
1·600000	0·4396	1·3241	0·5281	2·507	0·5095	1·822975	0·0015	2·4045	0·6843	3·514	0·9738
1·619962	0·4142	1·3628	0·5385	2·531	0·5353	1·822981	0·0007	2·4074	0·6843	3·518	0·9737
1·630000	0·4011	1·3836	0·5438	2·544	0·5490	1·822983	0·0004	2·4087	0·6843	3·520	0·9738
1·650000	0·3740	1·4279	0·5551	2·572	0·5778	1·822983	0·0002	2·4099	0·6843	3·521	0·9738

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and on substituting for x_S from (257) this reduces to

$$\frac{1\cdot4397}{(1\cdot822983-x)^{\frac{1}{2}}}\left(\frac{\partial u}{\partial y}\right)_{y=0}=\alpha_1+1\cdot267\alpha_1^2(1\cdot822983-x)^{\frac{1}{4}}$$
$$+1\cdot683\alpha_1^3(1\cdot822983-x)^{\frac{1}{2}}+2\cdot589\alpha_1^4(1\cdot822983-x)^{\frac{3}{4}}+\ldots. \quad (259)$$

In table 9 the second column gives the value of the left-hand side of equation (259) and the succeeding columns give values of the right-hand side for various α_1.

TABLE 9

x (radians)	left-hand side of (259)	right-hand side of (259) $\alpha_1=0\cdot6$	$\alpha_1=0\cdot7$	$\alpha_1=0\cdot8$	$\alpha_1=0\cdot65$	$\alpha_1=0\cdot67$	$\alpha_1=0\cdot68$
1·794	1·022	0·874	1·098	1·356	0·982	1·027	1·051
1·8	0·996	0·852	1·066	1·309	0·956	0·999	1·021
1·805	0·971	0·832	1·035	1·264	0·931	0·972	0·993
1·809	0·949	0·813	1·007	1·224	0·907	0·947	0·966
1·813	0·920	0·791	0·974	1·176	0·880	0·917	0·936
1·816	0·895	0·770	0·943	1·132	0·854	0·889	0·907
1·819	0·860	0·743	0·902	1·075	0·821	0·853	0·869
1·821	0·826	0·716	0·863	1·019	0·788	0·817	0·832
1·822	0·802	0·694	0·831	0·976	0·762	0·789	0·803
1·82245	0·781	0·679	0·810	0·947	0·744	0·770	0·783

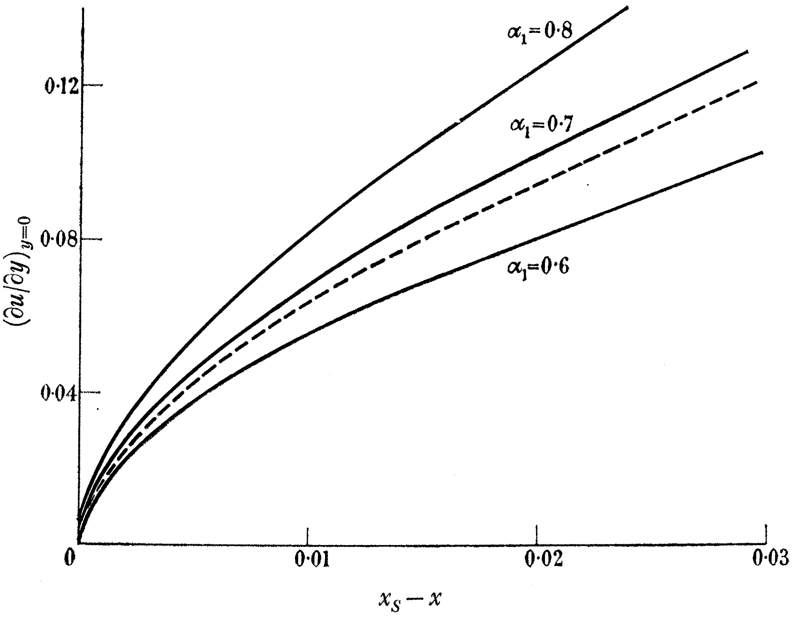

FIGURE 2. Variation of the skin friction with the distance from the separation point. ---, numerical results; ——, asymptotic solutions.

The first term that has been neglected in (259) is

$$\alpha_1^5(A_5-0\cdot26\ln x_1)\,(1\cdot822983-x), \quad (260)$$

where A_5 is a constant. Thus α_1 was determined by ensuring that the difference between the left- and right-hand sides of equation (259) was of the same order as (260). This gave

$$\alpha_1=0\cdot677\pm0\cdot003. \quad (261)$$

In figure 2 $(\partial u/\partial y)_{y=0}$ has been plotted against $(x_S - x)$ for the numerical results and for equation (259) when $\alpha_1 = 0{\cdot}6$, $0{\cdot}7$ and $0{\cdot}8$. The agreement between the numerical results and the asymptotic solution near separation is clearly shown.

Finally the velocity profiles near separation given by the numerical results and by the asymptotic solution will be compared. From equations (181), (182) and (132) the velocity distribution close to separation is

$$u_1 = (\partial f/\partial \eta)_{x=x_S} = \tfrac{1}{2}y_1^2 - \tfrac{1}{6}\alpha_1^2 y_1^4 - 0{\cdot}1352\alpha_1^3 y_1^5 - \{0{\cdot}0596\alpha_1^4 + \tfrac{1}{360}(1-\tan^2 x_S)\} y_1^6 - 0{\cdot}00089\alpha_1^5 y_1^7, \quad (262)$$

where, from equation (253),

$$y_1 = \{\sec^2 \tfrac{1}{2}x_S - 2\}^{\frac{1}{2}}\,\eta = 0{\cdot}815\eta. \quad (263)$$

For $\eta \leqslant 1$, $x_1 = 10^{-6}$ and $\alpha_1 = 0{\cdot}68$ it is easily shown that the term (252) is negligible. In table 10 the velocity profile obtained numerically and the velocity profiles for different α_1

TABLE 10. THE VELOCITY DISTRIBUTION NEAR SEPARATION

η	numerical results	$\alpha_1 = 0{\cdot}67$	$\alpha_1 = 0{\cdot}68$	$\alpha_1 = 0{\cdot}69$
0·05	0·0008	0·0008	0·0008	0·0008
0·10	0·0033	0·0033	0·0033	0·0033
0·15	0·0075	0·0075	0·0075	0·0075
0·20	0·0132	0·0132	0·0132	0·0132
0·25	0·0206	0·0206	0·0206	0·0206
0·30	0·0296	0·0296	0·0296	0·0296
0·35	0·0402	0·0402	0·0402	0·0401
0·40	0·0522	0·0522	0·0522	0·0522
0·45	0·0657	0·0658	0·0657	0·0657
0·50	0·0807	0·0807	0·0806	0·0805
0·55	0·0969	0·0970	0·0969	0·0968
0·60	0·1145	0·1146	0·1144	0·1143
0·65	0·1333	0·1335	0·1332	0·1329
0·70	0·1532	0·1534	0·1531	0·1527
0·75	0·1742	0·1745	0·1740	0·1735
0·80	0·1961	0·1965	0·1958	0·1951
0·85	0·2190	0·2194	0·2186	0·2176
0·90	0·2425	0·2432	0·2420	0·2408
0·95	0·2667	0·2676	0·2661	0·2646
1·00	0·2916	0·2927	0·2908	0·2889

computed from equations (262) and (263) are given. It can be seen that there is excellent agreement between the velocity profile obtained numerically and those obtained from the asymptotic expansion.

Comparison between these profiles gives

$$\alpha_1 = 0{\cdot}676 \pm 0{\cdot}002, \quad (264)$$

which confirms the value (261).

16. THE EQUATION OF SIMILAR PROFILES

A range of solutions of the equation of similar profiles have been obtained by the author on the Manchester University Mark I computer. It has been noted earlier that a solution of the equation of similar profiles is required for the distribution of $\partial f/\partial \eta$ at the leading edge.

In particular, solutions of

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \left\{1 - \left(\frac{df}{d\eta}\right)^2\right\} = 0, \quad (265)$$

with boundary conditions

$$\left.\begin{array}{lll}\text{at} & \eta = 0, & f(0) = v_0, \quad \mathrm{d}f/\mathrm{d}\eta = 0, \\ \text{as} & \eta \to \infty, & \mathrm{d}f/\mathrm{d}\eta \to 1,\end{array}\right\} \tag{266}$$

were obtained for several values of v_0. These solutions are given in table 11.

Table 11

$f(0)$	+10	+5	+4	+3	+2	+1	+0·5	0	−1	−2
$f''(0)$ \ η	10·193961	5·3595396	4·4289466	3·5266403	2·6700560	1·8893138	1·5417511	1·2325877	0·7565749	0·4758106
0·05	0·4001	0·2360	0·1995	0·1626	0·1258	0·0909	0·0749	0·0604	0·0375	0·0237
0·10	0·6409	0·4176	0·3607	0·3002	0·2373	0·1750	0·1455	0·1183	0·0744	0·0473
0·15	0·7856	0·5571	0·4906	0·4164	0·3359	0·2525	0·2119	0·1737	0·1106	0·0708
0·20	0·8723	0·6639	0·5949	0·5144	0·4229	0·3239	0·2742	0·2266	0·1461	0·0941
0·25	0·9241	0·7456	0·6787	0·5968	0·4995	0·3896	0·3327	0·2771	0·1810	0·1172
0·30	0·9550	0·8078	0·7456	0·6659	0·5668	0·4498	0·3874	0·3252	0·2150	0·1401
0·35	0·9734	0·8552	0·7991	0·7238	0·6257	0·5049	0·4384	0·3710	0·2484	0·1629
0·40	0·9843	0·8911	0·8417	0·7721	0·6773	0·5553	0·4860	0·4145	0·2809	0·1855
0·45	0·9908	0·9183	0·8755	0·8124	0·7223	0·6012	0·5304	0·4557	0·3127	0·2078
0·50	0·9946	0·9389	0·9024	0·8459	0·7616	0·6431	0·5715	0·4946	0·3436	0·2299
0·6	0·9981	0·9660	0·9403	0·8967	0·8252	0·7155	0·6451	0·5663	0·4031	0·2735
0·7	0·9994	0·9813	0·9638	0·9313	0·8729	0·7749	0·7079	0·6299	0·4593	0·3160
0·8	0·9998	0·9898	0·9783	0·9547	0·9083	0·8231	0·7611	0·6859	0·5121	0·3575
0·9	0·9999	0·9945	0·9871	0·9704	0·9344	0·8620	0·8060	0·7351	0·5615	0·3979
1·0	1·0000	0·9970	0·9924	0·9808	0·9534	0·8931	0·8435	0·7779	0·6075	0·4370
1·2	1·0000	0·9992	0·9974	0·9922	0·9771	0·9373	0·9002	0·8467	0·6896	0·5115
1·4	1·0000	0·9998	0·9992	0·9969	0·9891	0·9644	0·9381	0·8968	0·7590	0·5806
1·6	—	0·9999	0·9997	0·9988	0·9950	0·9803	0·9627	0·9323	0·8164	0·6440
1·8	—	1·0000	0·9999	0·9996	0·9978	0·9895	0·9782	0·9568	0·8629	0·7015
2·0	—	1·0000	1·0000	0·9998	0·9991	0·9946	0·9876	0·9732	0·8998	0·7530
2·4	—	1·0000	1·0000	1·0000	0·9998	0·9987	0·9964	0·9905	0·9500	0·8381
2·8	—	—	1·0000	1·0000	1·0000	0·9997	0·9991	0·9970	0·9774	0·9006
3·2	—	—	—	1·0000	1·0000	0·9999	0·9998	0·9992	0·9908	0·9434
3·6	—	—	—	—	1·0000	1·0000	1·0000	0·9998	0·9967	0·9703
4·0	—	—	—	—	—	1·0000	1·0000	1·0000	0·9989	0·9858
4·6	—	—	—	—	—	1·0000	1·0000	1·0000	0·9998	0·9962
5·2	—	—	—	—	—	—	—	1·0000	1·0000	0·9992
5·8	—	—	—	—	—	—	—	—	1·0000	0·9999
6·4	—	—	—	—	—	—	—	—	1·0000	1·0000

Also solutions of
$$\frac{\mathrm{d}^3f}{\mathrm{d}\eta^3} + f\,\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2} + \beta\left\{1 - \left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\right\} = 0 \tag{267}$$

for $\beta < 0$ and with boundary conditions:

$$\left.\begin{array}{lll}\text{at} & \eta = 0, & \mathrm{d}f/\mathrm{d}\eta = \mathrm{d}^2f/\mathrm{d}\eta^2 = 0, \\ \text{as} & \eta \to \infty, & \mathrm{d}f/\mathrm{d}\eta \to 1,\end{array}\right\} \tag{268}$$

were considered. The values obtained for $f(0)$ corresponding to selected values of β are given in table 12 and also results given by various others have been included. In figure 3, $f(0)$ has been plotted against β giving the curve which divides the wholly forward flows from flows with backflow. It may be conjectured that for $|\beta|$ large this curve is asymptotically of the form

$$f(0) \sim 2^{\frac{3}{2}}(|\beta|)^{\frac{1}{2}} \tag{269}$$

and the results confirm this.

It is interesting to compare the velocity distribution at separation with a corresponding solution of the equation of similar profiles. We introduce the suction parameter σ where

$$\sigma = (U_0/\nu l)^{\frac{1}{2}}\,\delta_2 v_S(x). \tag{270}$$

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Since $v_S(x)$ is non-dimensional so is σ. For the equation of similar profiles

$$\sigma = f(0)\,\Delta_2, \tag{271}$$

where

$$\Delta_2 = \int_0^\infty \frac{df}{d\eta}\left(1-\frac{df}{d\eta}\right)d\eta. \tag{272}$$

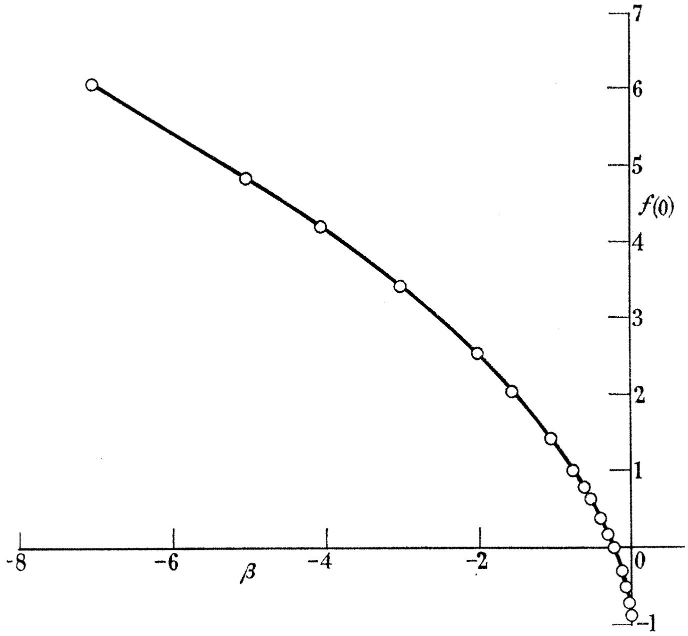

FIGURE 3

TABLE 12

β	$f(0)$	β	$f(0)$	
−0·05	−0·5008	0	−0·8757	Emmons & Leigh (1953)
−0·1	−0·2997	−0·0145	−0·7046	Brown & Donoughe (1951)
−0·198838	0	−0·0872	−0·3461	
−0·5	+0·6460	−0·287	+0·2	Thwaites (1949)
−1·0	+1·4142	−0·371	+0·4	
−1·5	+2·0245	−0·474	+0·6	
−2·0	+2·5489	−0·592	+0·8	
−3	+3·4429	−0·721	+1	
−4	+4·2062	−1·0	$2^{\frac{1}{2}}$	Thwaites (1949)
−5	+4·883			
−7	+6·065			
−10	+7·561			
−18	+10·69			

The velocity profiles will be compared for the same value of σ so that

$$f(0)\,\Delta_2 = (U_0/\nu l)^{\frac{1}{2}}\,\delta_2 v_S(x). \tag{273}$$

For the flow without suction $f(0) = 0$ and therefore the required profile is given by the solution of equation (267) with boundary conditions (268) for $\beta = -0{\cdot}198838$. This is shown in figure 4 plotted against η/Δ_2 together with the separation profile plotted against y/δ_2, where the momentum thickness has been taken as an appropriate measure of the boundary-layer thickness.

From table 8, for the separation profile

$$H = 3\cdot521, \tag{274}$$

whereas for the solution of the equation of similar profiles

$$H = 4\cdot030. \tag{275}$$

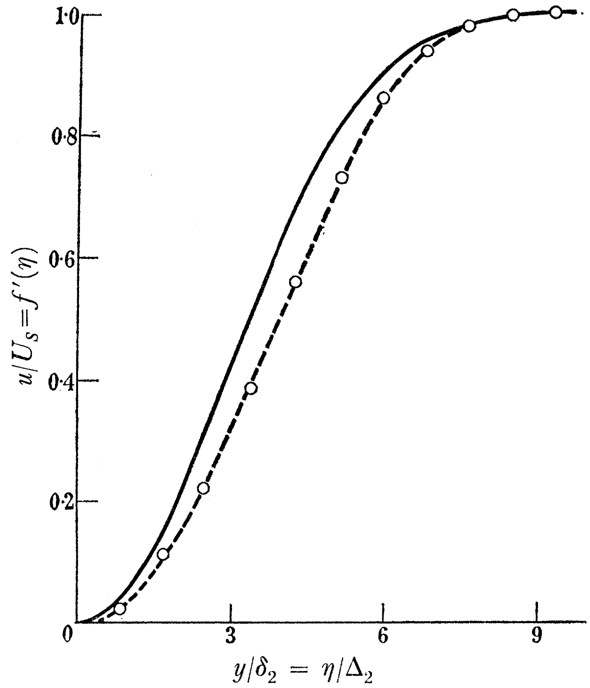

FIGURE 4. Comparison between the velocity distribution at separation and a corresponding solution of the equation of similar profiles for $v_s(x) = 0$. ——, separation profile; -- ○--, $\beta = -0\cdot198838$.

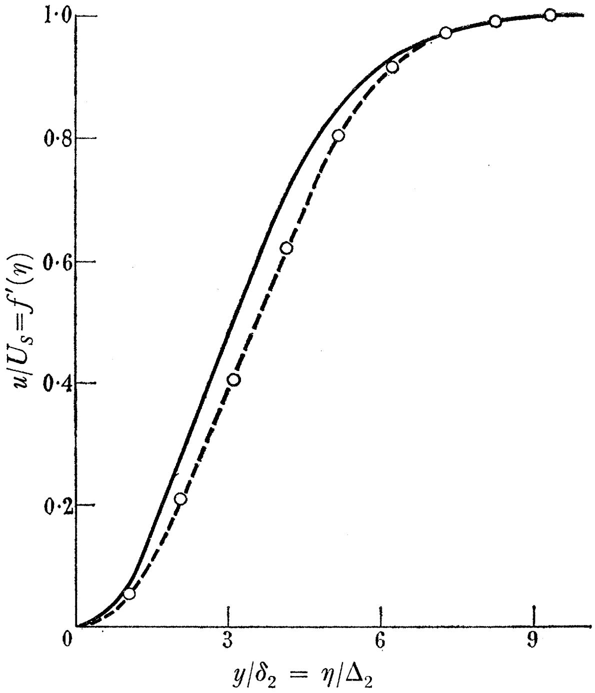

FIGURE 5. Comparison between the velocity distribution at separation and a corresponding solution of the equation of similar profiles for $v_s(x) = 0\cdot5$. ——, separation profile; -- ○--, $\beta = -0\cdot52$.

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In the case of flow with constant suction $v_s(x) = 0{\cdot}5$ equation (273) gives

$$\sigma = f(0)\,\Delta_2 = 0{\cdot}3242. \tag{276}$$

By interpolating between the solutions given in table 12 it is found that $\beta = -0{\cdot}52$ and also the required velocity profile can be obtained. This is shown in figure 5, together with the separation profile.

From table 4, for the separation profile

$$H = 3{\cdot}291, \tag{277}$$

whereas for the solution of the equation of similar profiles

$$H = 3{\cdot}639. \tag{278}$$

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